bas.1 The Theory of a Structure

Every structure $M$ makes some sentences true, and some false. The set of all the sentences it makes true is called its theory. That set is in fact a theory, since anything it entails must be true in all its models, including $M$.

**Definition bas.1.** Given a structure $M$, the theory of $M$ is the set $\text{Th}(M)$ of sentences that are true in $M$, i.e., $\text{Th}(M) = \{ \varphi : M \models \varphi \}$.

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

**Proposition bas.2.** For any $M$, $\text{Th}(M)$ is complete.

*Proof.* For any sentence $\varphi$ either $M \models \varphi$ or $M \models \neg \varphi$, so either $\varphi \in \text{Th}(M)$ or $\neg \varphi \notin \text{Th}(M)$. □

**Proposition bas.3.** If $M \models \varphi$ for every $\varphi \in \text{Th}(M)$, then $M \equiv N$.

*Proof.* Since $M \models \varphi$ for all $\varphi \in \text{Th}(M)$, $\text{Th}(M) \subseteq \text{Th}(N)$. If $M \models \varphi$, then $M \not\models \neg \varphi$, so $\neg \varphi \notin \text{Th}(M)$. Since $\text{Th}(M)$ is complete, $\varphi \in \text{Th}(M)$. So, $\text{Th}(M) \subseteq \text{Th}(N)$, and we have $M \equiv N$. □

**Remark 1.** Consider $\mathfrak{R} = (\mathbb{R}, <)$, the structure whose domain is the set $\mathbb{R}$ of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the $<$ relation over the reals. Clearly $\mathfrak{R}$ is non-enumerable; however, since $\text{Th}(\mathfrak{R})$ is obviously consistent, by the Löwenheim–Skolem theorem it has an enumerable model, say $\mathfrak{S}$, and by Proposition bas.3, $\mathfrak{R} \equiv \mathfrak{S}$. Moreover, since $\mathfrak{R}$ and $\mathfrak{S}$ are not isomorphic, this shows that the converse of ?? fails in general.

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Bibliography