

bas.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as **structures** for these languages. The most common case is when all the symbols in a **language** \mathcal{L} are also part of a **language** \mathcal{L}' , i.e., $\mathcal{L} \subseteq \mathcal{L}'$. An \mathcal{L} -**structure** \mathfrak{M} can then always be expanded to an \mathcal{L}' -**structure** by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an \mathcal{L}' -structure \mathfrak{M}' we can obtain an \mathcal{L} -structure simply by “forgetting” the interpretations of the symbols that do not occur in \mathcal{L} .

mod:bas:red:
defn:reduct **Definition bas.1.** Suppose $\mathcal{L} \subseteq \mathcal{L}'$, \mathfrak{M} is an \mathcal{L} -**structure** and \mathfrak{M}' is an \mathcal{L}' -**structure**. \mathfrak{M} is the *reduct* of \mathfrak{M}' to \mathcal{L} , and \mathfrak{M}' is an *expansion* of \mathfrak{M} to \mathcal{L}' iff

1. $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every **constant symbol** $c \in \mathcal{L}$, $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$.
3. For every **function symbol** $f \in \mathcal{L}$, $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$.
4. For every **predicate symbol** $P \in \mathcal{L}$, $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$.

mod:bas:red:
prop:reduct **Proposition bas.2.** If an \mathcal{L} -**structure** \mathfrak{M} is a *reduct* of an \mathcal{L}' -**structure** \mathfrak{M}' , then for all \mathcal{L} -*sentences* φ ,

$$\mathfrak{M} \models \varphi \text{ iff } \mathfrak{M}' \models \varphi.$$

Proof. Exercise. □

Problem bas.1. Prove **Proposition bas.2**.

Definition bas.3. When we have an \mathcal{L} -structure \mathfrak{M} , and $\mathcal{L}' = \mathcal{L} \cup \{P\}$ is the expansion of \mathcal{L} obtained by adding a single n -place **predicate symbol** P , and $R \subseteq |\mathfrak{M}|^n$ is an n -place relation, then we write (\mathfrak{M}, R) for the expansion \mathfrak{M}' of \mathfrak{M} with $P^{\mathfrak{M}'} = R$.

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Bibliography