bas.1 Partial Isomorphisms

Definition bas.1. Given two structures $\mathfrak{M}$ and $\mathfrak{N}$, a *partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$ is a finite partial function p taking arguments in $|\mathfrak{M}|$ and returning values in $|\mathfrak{N}|$, which satisfies the isomorphism conditions from **??** on its domain:

1. p is injective;
2. for every constant symbol c: if $p(c^{\mathfrak{M}})$ is defined, then $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$;
3. for every n-place predicate symbol P: if $a_1, \ldots, a_n$ are in the domain of p, then $\langle a_1, \ldots, a_n \rangle \in P^{\mathfrak{M}}$ if and only if $\langle p(a_1), \ldots, p(a_n) \rangle \in P^{\mathfrak{N}}$;
4. for every n-place function symbol f: if $a_1, \ldots, a_n$ are in the domain of p, then $p(f^{\mathfrak{M}}(a_1, \ldots, a_n)) = f^{\mathfrak{N}}(p(a_1),\ dots, p(a_n))$.

That p is finite means that $\operatorname{dom}(p)$ is finite.

Notice that the empty function $\emptyset$ is always a partial isomorphism between any two structures.

mod:bas:pis: defn:partialisom **Definition bas.2.** Two structures $\mathfrak{M}$ and $\mathfrak{N}$, are *partially isomorphic*, written $\mathfrak{M} \simeq_p \mathfrak{N}$, if and only if there is a non-empty set I of partial isomorphisms between $\mathfrak{M}$ and $\mathfrak{N}$ satisfying the *back-and-forth* property:

1. (*Forth*) For every $p \in I$ and $a \in |\mathfrak{M}|$ there is $q \in I$ such that $p \subseteq q$ and a is in the domain of q;
2. (*Back*) For every $p \in I$ and $b \in |\mathfrak{N}|$ there is $q \in I$ such that $p \subseteq q$ and b is in the range of q.

mod:bas:pis: thm:p-isom1 **Theorem bas.3.** *If $\mathfrak{M} \simeq_p \mathfrak{N}$ and $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, then $\mathfrak{M} \simeq \mathfrak{N}$.*

Proof. Since $\mathfrak{M}$ and $\mathfrak{N}$ are enumerable, let $|\mathfrak{M}| = \{a_0, a_1, \ldots\}$ and $|\mathfrak{N}| = \{b_0, b_1, \ldots\}$. Starting with an arbitrary $p_0 \in I$, we define an increasing sequence of partial isomorphisms $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$ as follows:

1. if $n+1$ is odd, say $n = 2r$, then using the Forth property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and a_r is in the domain of p_{n+1};
2. if $n+1$ is even, say $n+1 = 2r$, then using the Back property find a $p_{n+1} \in I$ such that $p_n \subseteq p_{n+1}$ and b_r is in the range of p_{n+1}.

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that p is a an isomorphism between $\mathfrak{M}$ and $\mathfrak{N}$. □

Problem bas.1. Show in detail that p as defined in Theorem bas.3 is in fact an isomorphism.

Theorem bas.4. *Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are structures for a purely relational language (a language containing only predicate symbols, and no function symbols or constants). Then if $\mathfrak{M} \simeq_p \mathfrak{N}$, also $\mathfrak{M} \equiv \mathfrak{N}$.* mod:bas:pis: thm:p-isom2

Proof. By induction on formulas, one shows that if $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are such that there is a partial isomorphism p mapping each a_i to b_i and $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ (for $i = 1, \dots, n$), then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$. The case for $n = 0$ gives $\mathfrak{M} \equiv \mathfrak{N}$. □

Remark 1. If function symbols are present, the previous result is still true, but one needs to consider the isomorphism induced by p between the substructure of $\mathfrak{M}$ generated by $a_1, \dots, a_n$ and the substructure of $\mathfrak{N}$ generated by $b_1, \dots, b_n$.

The previous result can be "broken down" into stages by establishing a connection between the number of nested quantifiers in a formula and how many times the relevant partial isomorphisms can be extended.

Definition bas.5. For any formula φ, the *quantifier rank* of φ, denoted by $\mathrm{qr}(\varphi) \in \mathbb{N}$, is recursively defined as the highest number of nested quantifiers in φ. Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are *n-equivalent*, written $\mathfrak{M} \equiv_n \mathfrak{N}$, if they agree on all sentences of quantifier rank less than or equal to n.

Proposition bas.6. *Let $\mathcal{L}$ be a finite purely relational language, i.e., a language containing finitely many predicate symbols and constant symbols, and no function symbols. Then for each $n \in \mathbb{N}$ there are only finitely many first-order sentences in the language $\mathcal{L}$ that have quantifier rank no greater than n, up to logical equivalence.* mod:bas:pis: prop:qr-finite

Proof. By induction on n. □

Definition bas.7. Given a structure $\mathfrak{M}$, let $|\mathfrak{M}|^{<\omega}$ be the set of all finite sequences over $|\mathfrak{M}|$. We use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to range over finite sequences of elements. If $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$ and $a \in |\mathfrak{M}|$, then $\mathbf{a}a$ represents the *concatenation* of $\mathbf{a}$ with a.

Definition bas.8. Given structures $\mathfrak{M}$ and $\mathfrak{N}$, we define relations $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$ between sequences of equal length, by recursion on n as follows:

1. $I_0(\mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ satisfy the same atomic formulas in $\mathfrak{M}$ and $\mathfrak{N}$; i.e., if $s_1(x_i) = a_i$ and $s_2(x_i) = b_i$ and φ is atomic with all variables among $x_1, \dots, x_n$, then $\mathfrak{M}, s_1 \vDash \varphi$ if and only if $\mathfrak{N}, s_2 \vDash \varphi$.

2. $I_{n+1}(\mathbf{a}, \mathbf{b})$ if and only if for every $a \in A$ there is a $b \in B$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and vice-versa.

Definition bas.9. Write $\mathfrak{M} \approx_n \mathfrak{N}$ if $I_n(\Lambda, \Lambda)$ holds of $\mathfrak{M}$ and $\mathfrak{N}$ (where Λ is the empty sequence).

mod:bas:pis: thm:b-n-f **Theorem bas.10.** *Let $\mathcal{L}$ be a purely relational language. Then $I_n(\mathbf{a}, \mathbf{b})$ implies that for every φ such that $\mathrm{qr}(\varphi) \leq n$, we have $\mathfrak{M}, \mathbf{a} \vDash \varphi$ if and only if $\mathfrak{N}, \mathbf{b} \vDash \varphi$ (where again $\mathbf{a}$ satisfies φ if any s such that $s(x_i) = a_i$ satisfies φ). Moreover, if $\mathcal{L}$ is finite, the converse also holds.*

Proof. The proof that $I_n(\mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than n is by an easy induction on φ. For the converse we proceed by induction on n, using Proposition bas.6, which ensures that for each n there are at most finitely many non-equivalent formulas of that quantifier rank.

For $n = 0$ the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that $I_0(\mathbf{a}, \mathbf{b})$.

For the $n + 1$ case, suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same formulas of quantifier rank no greater than $n + 1$; in order to show that $I_{n+1}(\mathbf{a}, \mathbf{b})$ suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $I_n(\mathbf{a}a, \mathbf{b}b)$, and by the inductive hypothesis again suffices to show that for each $a \in |\mathfrak{M}|$ there is a $b \in |\mathfrak{N}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n.

Given $a \in |\mathfrak{M}|$, let τ_n^a be set of formulas $\psi(x, \mathbf{y})$ of rank no greater than n satisfied by $\mathbf{a}a$ in $\mathfrak{M}$; τ_n^a is finite, so we can assume it is a single first-order formula. It follows that $\mathbf{a}$ satisfies $\exists x\, \tau_n^a(x, \mathbf{y})$, which has quantifier rank no greater than $n + 1$. By hypothesis $\mathbf{b}$ satisfies the same formula in $\mathfrak{N}$, so that there is a $b \in |\mathfrak{N}|$ such that $\mathbf{b}b$ satisfies τ_n^a; in particular, $\mathbf{b}b$ satisfies the same formulas of quantifier rank no greater than n as $\mathbf{a}a$. Similarly one shows that for every $b \in |\mathfrak{N}|$ there is $a \in |\mathfrak{M}|$ such that $\mathbf{a}a$ and $\mathbf{b}b$ satisfy the same formulas of quantifier rank no greater than n, which completes the proof. □

mod:bas:pis: cor:b-n-f **Corollary bas.11.** *If $\mathfrak{M}$ and $\mathfrak{N}$ are purely relational structures in a finite language, then $\mathfrak{M} \approx_n \mathfrak{N}$ if and only if $\mathfrak{M} \equiv_n \mathfrak{N}$. In particular $\mathfrak{M} \equiv \mathfrak{N}$ if and only if for each n, $\mathfrak{M} \approx_n \mathfrak{N}$.*

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Bibliography