bas.1 Non-standard Models of Arithmetic

Definition bas.1. Let $\mathcal{L}_N$ be the language of arithmetic, comprising a constant symbol o, a 2-place predicate symbol $<$, a 1-place function symbol $\prime$, and 2-place function symbols $+$ and $\times$.

1. The *standard model* of arithmetic is the structure $\mathfrak{N}$ for $\mathcal{L}_N$ having $\mathbb{N} = \{0, 1, 2, \dots\}$ and interpreting o as 0, $<$ as the less-than relation over $\mathbb{N}$, and $\prime$, $+$ and $\times$ as successor, addition, and multiplication over $\mathbb{N}$, respectively.

2. *True arithmetic* is the theory $\mathrm{Th}(\mathfrak{N})$.

When working in $\mathcal{L}_N$ we abbreviate each term of the form $\mathsf{o}^{\prime\cdots\prime}$, with n applications of the successor function to o, as $\overline{n}$.

Definition bas.2. A structure $\mathfrak{M}$ for $\mathcal{L}_N$ is *standard* if and only $\mathfrak{N} \simeq \mathfrak{M}$.

mod:bas:nsa:thm:non-std **Theorem bas.3.** *There are non-standard enumerable models of true arithmetic.*

Proof. Expand $\mathcal{L}_N$ by introducing a new constant symbol c, and consider the theory

$$\mathrm{Th}(\mathfrak{N}) \cup \{\overline{n} < c : n \in \mathbb{N}\}.$$

The theory is finitely satisfiable, so by compactness it has a model $\mathfrak{M}$, which can be taken to be enumerable by the Downward Löwenheim-Skolem theorem. Where $|\mathfrak{M}|$ is the domain of $\mathfrak{M}$, let $\mathfrak{M}$ interpret the non-logical constants of $\mathcal{L}$ as $\mathbf{z} = \mathsf{o}^{\mathfrak{M}} \in |\mathfrak{M}|$, $\prec = <^{\mathfrak{M}} \subseteq M^2$, $* = \prime^{\mathfrak{M}} \colon |\mathfrak{M}| \to |\mathfrak{M}|$, and $\oplus = +^{\mathfrak{M}}, \otimes = \times^{\mathfrak{M}} \colon |\mathfrak{M}|^2 \to |\mathfrak{M}|$. For each $x \in |\mathfrak{M}|$, we write x^* for the element of $|\mathfrak{M}|$ obtained from x by application of $*$.

Now, if h were an isomorphism of $\mathfrak{N}$ and $\mathfrak{M}$, there would be $n \in \mathbb{N}$ such that $h(n) = c^{\mathfrak{M}}$. So let s be any assignment in $\mathfrak{N}$ such that $s(x) = n$. Then $\mathfrak{N}, s \models \overline{n} = x$; by the proof of **??**, also $\mathfrak{M}, h \circ s \models \overline{n} = x$, so that $c^{\mathfrak{M}} = \mathbf{z}^{*\cdots*}$ (with $*$ iterated n times). But this is impossible since by assumption $\mathfrak{M} \models \overline{n} < c$ and $\prec$ is irreflexive. So $\mathfrak{M}$ is non-standard. □

Problem bas.1. A relation R over a set X is *well-founded* if and only if there are no infinite descending chains in R, i.e., if there are no x_0, x_1, x_2, ... in X such that $\dots x_2 R x_1 R x_0$. Assuming Zermelo-Fraenkel set theory ZF is consistent, show that there are non-well-founded models of ZF, i.e., models $\mathfrak{M}$ such that $\dots x_2 \in x_1 \in x_0$.

Since the non-standard model $\mathfrak{M}$ from Theorem bas.3 is elementarily equivalent to the standard one, a number of properties of $\mathfrak{M}$ can be derived. The rest of this section is devoted to such a task, which will allow us to obtain a precise characterization of enumerable non-standard models of $\mathrm{Th}(\mathfrak{N})$.

1. No member of $|\mathfrak{M}|$ is $\prec$-less than itself: the sentence $\forall x\, \neg x < x$ is true in $\mathfrak{N}$ and therefore in $\mathfrak{M}$.

2. By a similar reasoning we obtain that $\prec$ is a *linear ordering* of $|\mathfrak{M}|$, i.e., a total, irreflexive, transitive relation on $|\mathfrak{M}|$.

3. The element $\mathbf{z}$ is the $\prec$-least element of $|\mathfrak{M}|$.

4. Any member of $|\mathfrak{M}|$ is $\prec$-less than its $*$-successor and x^* is the $\prec$-least member of $|\mathfrak{M}|$ greater than x.

5. $\mathfrak{M}$ contains an initial segment (of $\prec$) isomorphic to $\mathbb{N}$: $\mathbf{z}, \mathbf{z}^*, \mathbf{z}^{**}, \dots$, which we call the *standard part* of $|\mathfrak{M}|$. Any other member of $|\mathfrak{M}|$ is *non-standard*. There must be non-standard members of $|\mathfrak{M}|$, or else the function h from the proof of Theorem bas.3 is an isomorphism. We use $n, m, \dots$ as variables ranging on this standard part of $\mathfrak{M}$.

6. Every non-standard element is greater than any standard one; this is because for every $n \in \mathbb{N}$,

$$\mathfrak{N} \models \forall z\,(\neg(z = \mathsf{o} \lor \dots \lor z = \overline{n}) \to \overline{n} < z),$$

   so if $z \in |\mathfrak{M}|$ is different from all the standard elements, it must be *greater* than all of them.

7. Any member of $|\mathfrak{M}|$ other than $\mathbf{z}$ is the $*$-successor of some unique element of $|\mathfrak{M}|$, denoted by ${}^*x$. If $x = y^*$ then both x and y are standard if one of them is (and both non-standard if one of them is).

8. Define an equivalence relation $\approx$ over $|\mathfrak{M}|$ by saying that $x \approx y$ if and only if for some *standard* n, either $x \oplus n = y$ or $y \oplus n = x$. In other words, $x \approx y$ if and only if x and y are a finite distance apart. If n and m are standard then $n \approx m$. Define the *block* of x to be the equivalence class $[x] = \{y : x \approx y\}$.

9. Suppose that $x \prec y$ where $x \not\approx y$. Since $\mathfrak{N} \models \forall x\,\forall y\,(x < y \to (x' < y \lor x' = y))$, either $x^* \prec y$ or $x^* = y$. The latter is impossible because it implies $x \approx y$, so $x \prec y$. Similarly, if $x \prec y$ and $x \not\approx y$, then $x \prec {}^*y$. Therefore if $x \prec y$ and $x \not\approx y$, then every $w \approx x$ is $\prec$-less than every $v \approx y$. Accordingly, each block $[x]$ forms a doubly infinite chain

$$\cdots \prec {}^{**}x \prec {}^*x \prec x \prec x^* \prec x^{**} \prec \cdots$$

   which is referred to as a *Z*-chain because it has the order type of the integers.

10. The $\prec$ ordering can be lifted up to the blocks: if $x \prec y$ then the block of x is less than the block of y. A block is *non-standard* if it contains a non-standard element. The standard block is the least block.

11. There is no least non-standard block: if y is non-standard then there is a $x \prec y$ where x is also non-standard and $x \not\approx y$. Proof: in the standard model $\mathfrak{N}$, every number is divisible by two, possibly with remainder one,

i.e., $\mathfrak{N} \models \forall y\, \forall x\, (y = x + x \lor y = x + x + \mathsf{o}')$. By elementary equivalence, for every $y \in |\mathfrak{M}|$ there is $x \in |\mathfrak{M}|$ such that either $x \oplus x = y$ or $x \oplus x \oplus \mathbf{z}^* = y$. If x were standard, then so would be y; so x is non-standard. Furthermore, x and y belong to different blocks, i.e, $x \not\approx y$. To see this, assume they did belong to the same block, i.e., $x \oplus n = y$ for some standard n. If $y = x \oplus x$, then $x \oplus n = x \oplus x$, whence $x = n$ by the cancellation law for addition (which holds in $\mathfrak{N}$ and therefore in $\mathfrak{M}$ as well), and x would be standard after all. Similarly if $y = x \oplus x \oplus \mathbf{z}^*$.

12. By a similar argument, there is no greatest block.

13. The ordering of the blocks is dense: if $[x]$ is less than $[y]$ (where $x \not\approx y$), then there is a block $[z]$ distinct from both that is between them. Suppose $x \prec y$. As before, $x \oplus y$ is divisible by two (possibly with remainder) so there is a $u \in |\mathfrak{M}|$ such that either $x \oplus y = u \oplus u$ or $x \oplus y = u \oplus u \oplus \mathbf{z}^*$. The element u is the average of x and y, and so is between them. Assume $x \oplus y = u \oplus u$ (the other case being similar): if $u \approx x$ then for some standard n:
$$x \oplus y = x \oplus n \oplus x \oplus n,$$
so $y = x \oplus n \oplus n$ and we would have $x \approx y$, against assumption. We conclude that $u \not\approx x$. A similar argument gives $u \not\approx y$.

The non-standard blocks are therefore ordered like the rationals: they form an enumerable linear ordering without endpoints. It follows that for any two enumerable non-standard models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of true arithmetic, their reducts to the language containing $<$ and $=$ only are isomorphic. Indeed, an isomorphism h can be defined as follows: the standard parts of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic to the standard model $\mathfrak{N}$ and hence to each other. The blocks making up the non-standard part are themselves ordered like the rationals and therefore by **??** are isomorphic; an isomorphism of the blocks can be extended to an isomorphism *within* the blocks by matching up arbitrary elements in each, and then taking the image of the successor of x in $\mathfrak{M}_1$ to be the successor of the image of x in $\mathfrak{M}_2$. Note that it does *not* follow that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic in the full language of arithmetic (indeed, isomorphism is always relative to a signature), as there are non-isomorphic ways to define addition and multiplication over $|\mathfrak{M}_1|$ and $|\mathfrak{M}_2|$. (This also follows from a famous theorem due to Vaught that the number of countable models of a complete theory cannot be 2.)

Problem bas.2. Show that there can be no greatest block in a non-standard model of arithmetic.

Problem bas.3. Let $\mathcal{L}$ be the first-order language containing $<$ as its only predicate symbol (besides $=$), and let $\mathfrak{N} = (\mathbb{N}, <)$. All the finite or cofinite subsets of $\mathfrak{N}$ are definable. Show that these are the *only* definable subsets of $\mathfrak{N}$.

(Hint: First, let $\mathit{prc}(x, y)$ be the $\mathcal{L}$-formula abbreviating "x is the immediate predecessor of y:"

$$x < y \land \neg\exists z\,(x < z \land z < y).$$

Now, to any definable subset of $\mathfrak{N}$ there corresponds a formula $\varphi(x)$ in $\mathcal{L}$. For any such φ, consider the sentence θ:

$$\exists x\,\forall y\,\forall z\,((x < y \land x < z \land \mathit{prc}(y, z) \land \varphi(y)) \to \varphi(z)).$$

Show that $\mathfrak{N} \models \theta$ if and only if the subset of $\mathfrak{N}$ defined by φ is either finite or cofinite.

Now, let $\mathfrak{M}$ be a non-standard model elementarily equivalent to $\mathfrak{N}$. If $a \in |\mathfrak{M}|$ is non-standard, let $b, c \in |\mathfrak{M}|$ be greater than a, and let b be the immediate predecessor of c. Then there is an automorphism h of $|\mathfrak{M}|$ such that $h(b) = c$ (why?). Therefore, if b satisfies φ, so does c (why?). It follows that θ is true in $\mathfrak{M}$, and hence also in $\mathfrak{N}$. But this implies that the subset of $\mathfrak{N}$ defined by φ is either finite or co-finite.

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Bibliography