

# Chapter udf

## Basics of Model Theory

### bas.1 Reducts and Expansions

Often it is useful or necessary to compare languages which have symbols in common, as well as **structures** for these languages. The most common case is when all the symbols in a **language**  $\mathcal{L}$  are also part of a **language**  $\mathcal{L}'$ , i.e.,  $\mathcal{L} \subseteq \mathcal{L}'$ . An  $\mathcal{L}$ -**structure**  $\mathfrak{M}$  can then always be expanded to an  $\mathcal{L}'$ -**structure** by adding interpretations of the additional symbols while leaving the interpretations of the common symbols the same. On the other hand, from an  $\mathcal{L}'$ -structure  $\mathfrak{M}'$  we can obtain an  $\mathcal{L}$ -structure simply by “forgetting” the interpretations of the symbols that do not occur in  $\mathcal{L}$ .

mod:bas:red:  
defn:reduct **Definition bas.1.** Suppose  $\mathcal{L} \subseteq \mathcal{L}'$ ,  $\mathfrak{M}$  is an  $\mathcal{L}$ -**structure** and  $\mathfrak{M}'$  is an  $\mathcal{L}'$ -**structure**.  $\mathfrak{M}$  is the *reduct* of  $\mathfrak{M}'$  to  $\mathcal{L}$ , and  $\mathfrak{M}'$  is an *expansion* of  $\mathfrak{M}$  to  $\mathcal{L}'$  iff

1.  $|\mathfrak{M}| = |\mathfrak{M}'|$
2. For every **constant symbol**  $c \in \mathcal{L}$ ,  $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$ .
3. For every **function symbol**  $f \in \mathcal{L}$ ,  $f^{\mathfrak{M}} = f^{\mathfrak{M}'}$ .
4. For every **predicate symbol**  $P \in \mathcal{L}$ ,  $P^{\mathfrak{M}} = P^{\mathfrak{M}'}$ .

mod:bas:red:  
prop:reduct **Proposition bas.2.** If an  $\mathcal{L}$ -**structure**  $\mathfrak{M}$  is a reduct of an  $\mathcal{L}'$ -**structure**  $\mathfrak{M}'$ , then for all  $\mathcal{L}$ -**sentences**  $\varphi$ ,

$$\mathfrak{M} \models \varphi \text{ iff } \mathfrak{M}' \models \varphi.$$

*Proof.* Exercise. □

**Problem bas.1.** Prove **Proposition bas.2**.

**Definition bas.3.** When we have an  $\mathcal{L}$ -structure  $\mathfrak{M}$ , and  $\mathcal{L}' = \mathcal{L} \cup \{P\}$  is the expansion of  $\mathcal{L}$  obtained by adding a single  $n$ -place **predicate symbol**  $P$ , and  $R \subseteq |\mathfrak{M}|^n$  is an  $n$ -place relation, then we write  $(\mathfrak{M}, R)$  for the expansion  $\mathfrak{M}'$  of  $\mathfrak{M}$  with  $P^{\mathfrak{M}'} = R$ .

## bas.2 Substructures

The **domain** of a **structure**  $\mathfrak{M}$  may be a subset of another  $\mathfrak{M}'$ . But we should obviously only consider  $\mathfrak{M}$  a “part” of  $\mathfrak{M}'$  if not only  $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$ , but  $\mathfrak{M}$  and  $\mathfrak{M}'$  “agree” in how they interpret the symbols of the language at least on the shared part  $|\mathfrak{M}|$ .

mod:bas:sub:  
sec

**Definition bas.4.** Given **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  for the same language  $\mathcal{L}$ , we say that  $\mathfrak{M}$  is a **substructure** of  $\mathfrak{M}'$ , and  $\mathfrak{M}'$  an **extension** of  $\mathfrak{M}$ , written  $\mathfrak{M} \subseteq \mathfrak{M}'$ , iff

mod:bas:sub:  
defn:substructure

1.  $|\mathfrak{M}| \subseteq |\mathfrak{M}'|$ ,
2. For each constant  $c \in \mathcal{L}$ ,  $c^{\mathfrak{M}} = c^{\mathfrak{M}'}$ ;
3. For each  $n$ -place **function symbol**  $f \in \mathcal{L}$   $f^{\mathfrak{M}}(a_1, \dots, a_n) = f^{\mathfrak{M}'}(a_1, \dots, a_n)$  for all  $a_1, \dots, a_n \in |\mathfrak{M}|$ .
4. For each  $n$ -place **predicate symbol**  $R \in \mathcal{L}$ ,  $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}}$  iff  $\langle a_1, \dots, a_n \rangle \in R^{\mathfrak{M}'}$  for all  $a_1, \dots, a_n \in |\mathfrak{M}|$ .

*Remark 1.* If the language contains no constant or **function symbols**, then any  $N \subseteq |\mathfrak{M}|$  determines a **substructure**  $\mathfrak{N}$  of  $\mathfrak{M}$  with **domain**  $|\mathfrak{N}| = N$  by putting  $R^{\mathfrak{N}} = R^{\mathfrak{M}} \cap N^n$ .

mod:bas:sub:  
rem:substructure

## bas.3 Overspill

**Theorem bas.5.** *If a set  $\Gamma$  of sentences has arbitrarily large finite models, then it has an infinite model.*

mod:bas:ove:  
sec  
mod:bas:ove:  
overspill

*Proof.* Expand the language of  $\Gamma$  by adding countably many new constants  $c_0, c_1, \dots$  and consider the set  $\Gamma \cup \{c_i \neq c_j : i \neq j\}$ . To say that  $\Gamma$  has arbitrarily large finite models means that for every  $m > 0$  there is  $n \geq m$  such that  $\Gamma$  has a model of cardinality  $n$ . This implies that  $\Gamma \cup \{c_i \neq c_j : i \neq j\}$  is finitely satisfiable. By compactness,  $\Gamma \cup \{c_i \neq c_j : i \neq j\}$  has a model  $\mathfrak{M}$  whose domain must be infinite, since it satisfies all inequalities  $c_i \neq c_j$ .  $\square$

**Proposition bas.6.** *There is no sentence  $\varphi$  of any first-order language that is true in a **structure**  $\mathfrak{M}$  if and only if the domain  $|\mathfrak{M}|$  of the **structure** is infinite.*

mod:bas:ove:  
inf-not-fo

*Proof.* If there were such a  $\varphi$ , its negation  $\neg\varphi$  would be true in all and only the finite **structures**, and it would therefore have arbitrarily large finite models but it would lack an infinite model, contradicting **Theorem bas.5**.  $\square$

## bas.4 Isomorphic Structures

mod:bas:iso:  
sec

First-order **structures** can be alike in one of two ways. One way in which the can be alike is that they make the same **sentences** true. We call such **structures** *elementarily equivalent*. But structures can be very different and still make the same **sentences** true—for instance, one can be **enumerable** and the other not. This is because there are lots of features of a **structure** that cannot be expressed in first-order languages, either because the language is not rich enough, or because of fundamental limitations of first-order logic such as the Löwenheim–Skolem theorem. So another, stricter, aspect in which **structures** can be alike is if they are fundamentally the same, in the sense that they only differ in the objects that make them up, but not in their structural features. A way of making this precise is by the notion of an *isomorphism*.

mod:bas:iso:  
defn:elem-equiv

**Definition bas.7.** Given two **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  for the same **language**  $\mathcal{L}$ , we say that  $\mathfrak{M}$  is *elementarily equivalent to*  $\mathfrak{M}'$ , written  $\mathfrak{M} \equiv \mathfrak{M}'$ , if and only if for every **sentence**  $\varphi$  of  $\mathcal{L}$ ,  $\mathfrak{M} \models \varphi$  iff  $\mathfrak{M}' \models \varphi$ .

mod:bas:iso:  
defn:isomorphism

**Definition bas.8.** Given two **structures**  $\mathfrak{M}$  and  $\mathfrak{M}'$  for the same **language**  $\mathcal{L}$ , we say that  $\mathfrak{M}$  is *isomorphic to*  $\mathfrak{M}'$ , written  $\mathfrak{M} \simeq \mathfrak{M}'$ , if and only if there is a function  $h: |\mathfrak{M}| \rightarrow |\mathfrak{M}'|$  such that:

1.  $h$  is **injective**: if  $h(x) = h(y)$  then  $x = y$ ;
2.  $h$  is **surjective**: for every  $y \in |\mathfrak{M}'|$  there is  $x \in |\mathfrak{M}|$  such that  $h(x) = y$ ;
3. for every **constant symbol**  $c$ :  $h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$ ;
4. for every  $n$ -place **predicate symbol**  $P$ :

mod:bas:iso:  
defn:iso-const  
mod:bas:iso:  
defn:iso-pred

$$\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}} \quad \text{iff} \quad \langle h(a_1), \dots, h(a_n) \rangle \in P^{\mathfrak{M}'};$$

mod:bas:iso:  
defn:iso-func

5. for every  $n$ -place **function symbol**  $f$ :

$$h(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{M}'}(h(a_1), \dots, h(a_n)).$$

mod:bas:iso:  
thm:isom

**Theorem bas.9.** *If  $\mathfrak{M} \simeq \mathfrak{M}'$  then  $\mathfrak{M} \equiv \mathfrak{M}'$ .*

*Proof.* Let  $h$  be an isomorphism of  $\mathfrak{M}$  onto  $\mathfrak{M}'$ . For any assignment  $s$ ,  $h \circ s$  is the composition of  $h$  and  $s$ , i.e., the assignment in  $\mathfrak{M}'$  such that  $(h \circ s)(x) = h(s(x))$ . By induction on  $t$  and  $\varphi$  one can prove the stronger claims:

- a.  $h(\text{Val}_s^{\mathfrak{M}}(t)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$ .
- b.  $\mathfrak{M}, s \models \varphi$  iff  $\mathfrak{M}', h \circ s \models \varphi$ .

The first is proved by induction on the complexity of  $t$ .

1. If  $t \equiv c$ , then  $\text{Val}_s^{\mathfrak{M}}(c) = c^{\mathfrak{M}}$  and  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(c) = c^{\mathfrak{M}'}$ . Thus,  $h(\text{Val}_s^{\mathfrak{M}}(t)) = h(c^{\mathfrak{M}}) = c^{\mathfrak{M}'}$  (by (3) of **Definition bas.8**) =  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(t)$ .

2. If  $t \equiv x$ , then  $\text{Val}_s^{\mathfrak{M}}(x) = s(x)$  and  $\text{Val}_{h \circ s}^{\mathfrak{M}'}(x) = h(s(x))$ . Thus,  $h(\text{Val}_s^{\mathfrak{M}}(x)) = h(s(x)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(x)$ .
3. If  $t \equiv f(t_1, \dots, t_n)$ , then

$$\begin{aligned}\text{Val}_s^{\mathfrak{M}}(t) &= f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n)) \quad \text{and} \\ \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)).\end{aligned}$$

The induction hypothesis is that for each  $i$ ,  $h(\text{Val}_s^{\mathfrak{M}}(t_i)) = \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_i)$ . So,

$$\begin{aligned}h(\text{Val}_s^{\mathfrak{M}}(t)) &= h(f^{\mathfrak{M}}(\text{Val}_s^{\mathfrak{M}}(t_1), \dots, \text{Val}_s^{\mathfrak{M}}(t_n))) \\ &= h(f^{\mathfrak{M}}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n))) && \text{(bas.1) } \text{mod:bas:iso:} \\ &= f^{\mathfrak{M}'}(\text{Val}_{h \circ s}^{\mathfrak{M}'}(t_1), \dots, \text{Val}_{h \circ s}^{\mathfrak{M}'}(t_n)) && \text{(bas.2) } \text{iso-1} \\ &= \text{Val}_{h \circ s}^{\mathfrak{M}'}(t) && \text{mod:bas:iso:} \\ & && \text{iso-2}\end{aligned}$$

Here, eq. (bas.1) follows by induction hypothesis and eq. (bas.2) by (5) of Definition bas.8.

Part (b) is left as an exercise.

If  $\varphi$  is a sentence, the assignments  $s$  and  $h \circ s$  are irrelevant, and we have  $\mathfrak{M} \models \varphi$  iff  $\mathfrak{M}' \models \varphi$ .  $\square$

**Problem bas.2.** Carry out the proof of (b) of Theorem bas.9 in detail. Make sure to note where each of the five properties characterizing isomorphisms of Definition bas.8 is used.

**Definition bas.10.** An *automorphism* of a structure  $\mathfrak{M}$  is an isomorphism of  $\mathfrak{M}$  onto itself.

**Problem bas.3.** Show that for any structure  $\mathfrak{M}$ , if  $X$  is a definable subset of  $\mathfrak{M}$ , and  $h$  is an automorphism of  $\mathfrak{M}$ , then  $X = \{h(x) : x \in X\}$  (i.e.,  $X$  is fixed under  $h$ ).

## bas.5 The Theory of a Structure

Every structure  $\mathfrak{M}$  makes some sentences true, and some false. The set of all the sentences it makes true is called its *theory*. That set is in fact a theory, since anything it entails must be true in all its models, including  $\mathfrak{M}$ .

**Definition bas.11.** Given a structure  $\mathfrak{M}$ , the *theory* of  $\mathfrak{M}$  is the set  $\text{Th}(\mathfrak{M})$  of sentences that are true in  $\mathfrak{M}$ , i.e.,  $\text{Th}(\mathfrak{M}) = \{\varphi : \mathfrak{M} \models \varphi\}$ .

We also use the term “theory” informally to refer to sets of sentences having an intended interpretation, whether deductively closed or not.

**Proposition bas.12.** For any  $\mathfrak{M}$ ,  $\text{Th}(\mathfrak{M})$  is complete.

*Proof.* For any sentence  $\varphi$  either  $\mathfrak{M} \models \varphi$  or  $\mathfrak{M} \models \neg\varphi$ , so either  $\varphi \in \text{Th}(\mathfrak{M})$  or  $\neg\varphi \in \text{Th}(\mathfrak{M})$ .  $\square$

mod:bas:thm:  
prop:equiv **Proposition bas.13.** *If  $\mathfrak{N} \models \varphi$  for every  $\varphi \in \text{Th}(\mathfrak{M})$ , then  $\mathfrak{M} \equiv \mathfrak{N}$ .*

*Proof.* Since  $\mathfrak{N} \models \varphi$  for all  $\varphi \in \text{Th}(\mathfrak{M})$ ,  $\text{Th}(\mathfrak{M}) \subseteq \text{Th}(\mathfrak{N})$ . If  $\mathfrak{N} \models \varphi$ , then  $\mathfrak{N} \not\models \neg\varphi$ , so  $\neg\varphi \notin \text{Th}(\mathfrak{N})$ . Since  $\text{Th}(\mathfrak{N})$  is complete,  $\varphi \in \text{Th}(\mathfrak{N})$ . So,  $\text{Th}(\mathfrak{N}) \subseteq \text{Th}(\mathfrak{M})$ , and we have  $\mathfrak{M} \equiv \mathfrak{N}$ .  $\square$

mod:bas:thm:  
remark:R *Remark 2.* Consider  $\mathfrak{R} = \langle \mathbb{R}, < \rangle$ , the structure whose domain is the set  $\mathbb{R}$  of the real numbers, in the language comprising only a 2-place predicate symbol interpreted as the  $<$  relation over the reals. Clearly  $\mathfrak{R}$  is non-enumerable; however, since  $\text{Th}(\mathfrak{R})$  is obviously consistent, by the Löwenheim–Skolem theorem it has an enumerable model, say  $\mathfrak{S}$ , and by Proposition bas.13,  $\mathfrak{R} \equiv \mathfrak{S}$ . Moreover, since  $\mathfrak{R}$  and  $\mathfrak{S}$  are not isomorphic, this shows that the converse of Theorem bas.9 fails in general.

## bas.6 Partial Isomorphisms

**Definition bas.14.** Given two structures  $\mathfrak{M}$  and  $\mathfrak{N}$ , a *partial isomorphism* from  $\mathfrak{M}$  to  $\mathfrak{N}$  is a finite partial function  $p$  taking arguments in  $|\mathfrak{M}|$  and returning values in  $|\mathfrak{N}|$ , which satisfies the isomorphism conditions from Definition bas.8 on its domain:

1.  $p$  is injective;
2. for every constant symbol  $c$ : if  $p(c^{\mathfrak{M}})$  is defined, then  $p(c^{\mathfrak{M}}) = c^{\mathfrak{N}}$ ;
3. for every  $n$ -place predicate symbol  $P$ : if  $a_1, \dots, a_n$  are in the domain of  $p$ , then  $\langle a_1, \dots, a_n \rangle \in P^{\mathfrak{M}}$  if and only if  $\langle p(a_1), \dots, p(a_n) \rangle \in P^{\mathfrak{N}}$ ;
4. for every  $n$ -place function symbol  $f$ : if  $a_1, \dots, a_n$  are in the domain of  $p$ , then  $p(f^{\mathfrak{M}}(a_1, \dots, a_n)) = f^{\mathfrak{N}}(p(a_1), \dots, p(a_n))$ .

That  $p$  is finite means that  $\text{dom}(p)$  is finite.

Notice that the empty function  $\emptyset$  is always a partial isomorphism between any two structures.

mod:bas:pis:  
defn:partialisom **Definition bas.15.** Two structures  $\mathfrak{M}$  and  $\mathfrak{N}$ , are *partially isomorphic*, written  $\mathfrak{M} \simeq_p \mathfrak{N}$ , if and only if there is a non-empty set  $I$  of partial isomorphisms between  $\mathfrak{M}$  and  $\mathfrak{N}$  satisfying the *back-and-forth* property:

1. (*Forth*) For every  $p \in I$  and  $a \in |\mathfrak{M}|$  there is  $q \in I$  such that  $p \subseteq q$  and  $a$  is in the domain of  $q$ ;
2. (*Back*) For every  $p \in I$  and  $b \in |\mathfrak{N}|$  there is  $q \in I$  such that  $p \subseteq q$  and  $b$  is in the range of  $q$ .

**Theorem bas.16.** *If  $\mathfrak{M} \simeq_p \mathfrak{N}$  and  $\mathfrak{M}$  and  $\mathfrak{N}$  are **enumerable**, then  $\mathfrak{M} \simeq \mathfrak{N}$ .* mod:bas:pis:  
thm:p-isom1

*Proof.* Since  $\mathfrak{M}$  and  $\mathfrak{N}$  are **enumerable**, let  $|\mathfrak{M}| = \{a_0, a_1, \dots\}$  and  $|\mathfrak{N}| = \{b_0, b_1, \dots\}$ . Starting with an arbitrary  $p_0 \in I$ , we define an increasing sequence of partial isomorphisms  $p_0 \subseteq p_1 \subseteq p_2 \subseteq \dots$  as follows:

1. if  $n + 1$  is odd, say  $n = 2r$ , then using the Forth property find a  $p_{n+1} \in I$  such that  $p_n \subseteq p_{n+1}$  and  $a_r$  is in the domain of  $p_{n+1}$ ;
2. if  $n + 1$  is even, say  $n + 1 = 2r$ , then using the Back property find a  $p_{n+1} \in I$  such that  $p_n \subseteq p_{n+1}$  and  $b_r$  is in the range of  $p_{n+1}$ .

If we now put:

$$p = \bigcup_{n \geq 0} p_n,$$

we have that  $p$  is a an isomorphism between  $\mathfrak{M}$  and  $\mathfrak{N}$ . □

**Problem bas.4.** Show in detail that  $p$  as defined in **Theorem bas.16** is in fact an isomorphism.

**Theorem bas.17.** *Suppose  $\mathfrak{M}$  and  $\mathfrak{N}$  are **structures** for a purely relational language (a language containing only **predicate symbols**, and no **function symbols** or **constants**). Then if  $\mathfrak{M} \simeq_p \mathfrak{N}$ , also  $\mathfrak{M} \equiv \mathfrak{N}$ .* mod:bas:pis:  
thm:p-isom2

*Proof.* By induction on **formulas**, one shows that if  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are such that there is a partial isomorphism  $p$  mapping each  $a_i$  to  $b_i$  and  $s_1(x_i) = a_i$  and  $s_2(x_i) = b_i$  (for  $i = 1, \dots, n$ ), then  $\mathfrak{M}, s_1 \models \varphi$  if and only if  $\mathfrak{N}, s_2 \models \varphi$ . The case for  $n = 0$  gives  $\mathfrak{M} \equiv \mathfrak{N}$ . □

*Remark 3.* If **function symbols** are present, the previous result is still true, but one needs to consider the isomorphism induced by  $p$  between the **substructure** of  $\mathfrak{M}$  generated by  $a_1, \dots, a_n$  and the **substructure** of  $\mathfrak{N}$  generated by  $b_1, \dots, b_n$ .

The previous result can be “broken down” into stages by establishing a connection between the number of nested quantifiers in a **formula** and how many times the relevant partial isomorphisms can be extended.

**Definition bas.18.** For any **formula**  $\varphi$ , the **quantifier rank** of  $\varphi$ , denoted by  $\text{qr}(\varphi) \in \mathbb{N}$ , is recursively defined as the highest number of nested quantifiers in  $\varphi$ . Two **structures**  $\mathfrak{M}$  and  $\mathfrak{N}$  are  **$n$ -equivalent**, written  $\mathfrak{M} \equiv_n \mathfrak{N}$ , if they agree on all sentences of quantifier rank less than or equal to  $n$ .

**Proposition bas.19.** *Let  $\mathcal{L}$  be a finite purely relational language, i.e., a language containing finitely many **predicate symbols** and **constant symbols**, and no **function symbols**. Then for each  $n \in \mathbb{N}$  there are only finitely many first-order sentences in the language  $\mathcal{L}$  that have quantifier rank no greater than  $n$ , up to logical equivalence.* mod:bas:pis:  
prop:qr-finite

*Proof.* By induction on  $n$ . □

**Definition bas.20.** Given a structure  $\mathfrak{M}$ , let  $|\mathfrak{M}|^{<\omega}$  be the set of all finite sequences over  $|\mathfrak{M}|$ . We use  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$  to range over finite sequences of elements. If  $\mathbf{a} \in |\mathfrak{M}|^{<\omega}$  and  $a \in |\mathfrak{M}|$ , then  $\mathbf{a}a$  represents the *concatenation* of  $\mathbf{a}$  with  $a$ .

**Definition bas.21.** Given structures  $\mathfrak{M}$  and  $\mathfrak{N}$ , we define relations  $I_n \subseteq |\mathfrak{M}|^{<\omega} \times |\mathfrak{N}|^{<\omega}$  between sequences of equal length, by recursion on  $n$  as follows:

1.  $I_0(\mathbf{a}, \mathbf{b})$  if and only if  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same atomic formulas in  $\mathfrak{M}$  and  $\mathfrak{N}$ ; i.e., if  $s_1(x_i) = a_i$  and  $s_2(x_i) = b_i$  and  $\varphi$  is atomic with all variables among  $x_1, \dots, x_n$ , then  $\mathfrak{M}, s_1 \models \varphi$  if and only if  $\mathfrak{N}, s_2 \models \varphi$ .
2.  $I_{n+1}(\mathbf{a}, \mathbf{b})$  if and only if for every  $a \in A$  there is a  $b \in B$  such that  $I_n(\mathbf{a}a, \mathbf{b}b)$ , and vice-versa.

**Definition bas.22.** Write  $\mathfrak{M} \approx_n \mathfrak{N}$  if  $I_n(A, A)$  holds of  $\mathfrak{M}$  and  $\mathfrak{N}$  (where  $A$  is the empty sequence).

*mod:bas:pis: thm:b-n-f* **Theorem bas.23.** *Let  $\mathcal{L}$  be a purely relational language. Then  $I_n(\mathbf{a}, \mathbf{b})$  implies that for every  $\varphi$  such that  $\text{qr}(\varphi) \leq n$ , we have  $\mathfrak{M}, \mathbf{a} \models \varphi$  if and only if  $\mathfrak{N}, \mathbf{b} \models \varphi$  (where again  $\mathbf{a}$  satisfies  $\varphi$  if any  $s$  such that  $s(x_i) = a_i$  satisfies  $\varphi$ ). Moreover, if  $\mathcal{L}$  is finite, the converse also holds.*

*Proof.* The proof that  $I_n(\mathbf{a}, \mathbf{b})$  implies that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same formulas of quantifier rank no greater than  $n$  is by an easy induction on  $\varphi$ . For the converse we proceed by induction on  $n$ , using Proposition bas.19, which ensures that for each  $n$  there are at most finitely many non-equivalent formulas of that quantifier rank.

For  $n = 0$  the hypothesis that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same quantifier-free formulas gives that they satisfy the same atomic ones, so that  $I_0(\mathbf{a}, \mathbf{b})$ .

For the  $n + 1$  case, suppose that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the same formulas of quantifier rank no greater than  $n + 1$ ; in order to show that  $I_{n+1}(\mathbf{a}, \mathbf{b})$  suffices to show that for each  $a \in |\mathfrak{M}|$  there is a  $b \in |\mathfrak{N}|$  such that  $I_n(\mathbf{a}a, \mathbf{b}b)$ , and by the inductive hypothesis again suffices to show that for each  $a \in |\mathfrak{M}|$  there is a  $b \in |\mathfrak{N}|$  such that  $\mathbf{a}a$  and  $\mathbf{b}b$  satisfy the same formulas of quantifier rank no greater than  $n$ .

Given  $a \in |\mathfrak{M}|$ , let  $\tau_n^a$  be set of formulas  $\psi(x, \mathbf{y})$  of rank no greater than  $n$  satisfied by  $\mathbf{a}a$  in  $\mathfrak{M}$ ;  $\tau_n^a$  is finite, so we can assume it is a single first-order formula. It follows that  $\mathbf{a}$  satisfies  $\exists x \tau_n^a(x, \mathbf{y})$ , which has quantifier rank no greater than  $n + 1$ . By hypothesis  $\mathbf{b}$  satisfies the same formula in  $\mathfrak{N}$ , so that there is a  $b \in |\mathfrak{N}|$  such that  $\mathbf{b}b$  satisfies  $\tau_n^a$ ; in particular,  $\mathbf{b}b$  satisfies the same formulas of quantifier rank no greater than  $n$  as  $\mathbf{a}a$ . Similarly one shows that for every  $b \in |\mathfrak{N}|$  there is a  $a \in |\mathfrak{M}|$  such that  $\mathbf{a}a$  and  $\mathbf{b}b$  satisfy the same formulas of quantifier rank no greater than  $n$ , which completes the proof. □

*mod:bas:pis: cor:b-n-f* **Corollary bas.24.** *If  $\mathfrak{M}$  and  $\mathfrak{N}$  are purely relational structures in a finite language, then  $\mathfrak{M} \approx_n \mathfrak{N}$  if and only if  $\mathfrak{M} \equiv_n \mathfrak{N}$ . In particular  $\mathfrak{M} \equiv \mathfrak{N}$  if and only if for each  $n$ ,  $\mathfrak{M} \approx_n \mathfrak{N}$ .*

## bas.7 Dense Linear Orders

**Definition bas.25.** A dense linear ordering without endpoints is a structure  $\mathfrak{M}$  for the language containing a single 2-place predicate symbol  $<$  satisfying the following sentences:

1.  $\forall x \neg x < x$ ;
2.  $\forall x \forall y \forall z (x < y \rightarrow (y < z \rightarrow x < z))$ ;
3.  $\forall x \forall y (x < y \vee x = y \vee y < x)$ ;
4.  $\forall x \exists y x < y$ ;
5.  $\forall x \exists y y < x$ ;
6.  $\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$ .

**Theorem bas.26.** Any two enumerable dense linear orderings without endpoints are isomorphic.

*mod:bas:dlo:  
thm:cantorQ*

*Proof.* Let  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  be enumerable dense linear orderings without endpoints, with  $<_1 = <^{\mathfrak{M}_1}$  and  $<_2 = <^{\mathfrak{M}_2}$ , and let  $\mathcal{I}$  be the set of all partial isomorphisms between them.  $\mathcal{I}$  is not empty since at least  $\emptyset \in \mathcal{I}$ . We show that  $\mathcal{I}$  satisfies the Back-and-Forth property. Then  $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$ , and the theorem follows by **Theorem bas.16**.

To show  $\mathcal{I}$  satisfies the Forth property, let  $p \in \mathcal{I}$  and let  $p(a_i) = b_i$  for  $i = 1, \dots, n$ , and without loss of generality suppose  $a_1 <_1 a_2 <_1 \dots <_1 a_n$ . Given  $a \in |\mathfrak{M}_1|$ , find  $b \in |\mathfrak{M}_2|$  as follows:

1. if  $a <_1 a_1$  let  $b \in |\mathfrak{M}_2|$  be such that  $b <_2 b_1$ ;
2. if  $a_n <_1 a$  let  $b \in |\mathfrak{M}_2|$  be such that  $b_n <_2 b$ ;
3. if  $a_i <_1 a <_1 a_{i+1}$  for some  $i$ , then let  $b \in |\mathfrak{M}_2|$  be such that  $b_i <_2 b <_2 b_{i+1}$ .

It is always possible to find a  $b$  with the desired property since  $\mathfrak{M}_2$  is a dense linear ordering without endpoints. Define  $q = p \cup \{(a, b)\}$  so that  $q \in \mathcal{I}$  is the desired extension of  $p$ . This establishes the Forth property. The Back property is similar. So  $\mathfrak{M}_1 \simeq_p \mathfrak{M}_2$ ; by **Theorem bas.16**,  $\mathfrak{M}_1 \simeq \mathfrak{M}_2$ .  $\square$

**Problem bas.5.** Complete the proof of **Theorem bas.26** by verifying that  $\mathcal{I}$  satisfies the Back property.

*Remark 4.* Let  $\mathfrak{S}$  be any enumerable dense linear ordering without endpoints. Then (by **Theorem bas.26**)  $\mathfrak{S} \simeq \mathfrak{Q}$ , where  $\mathfrak{Q} = (\mathbb{Q}, <)$  is the enumerable dense linear ordering having the set  $\mathbb{Q}$  of the rational numbers as its domain. Now consider again the structure  $\mathfrak{R} = (\mathbb{R}, <)$  from **Remark 2**. We saw that there is an enumerable structure  $\mathfrak{S}$  such that  $\mathfrak{R} \equiv \mathfrak{S}$ . But  $\mathfrak{S}$  is an enumerable



dense linear ordering without endpoints, and so it is isomorphic (and hence elementarily equivalent) to the **structure**  $\mathfrak{Q}$ . By transitivity of elementary equivalence,  $\mathfrak{R} \equiv \mathfrak{Q}$ . (We could have shown this directly by establishing  $\mathfrak{R} \simeq_p \mathfrak{Q}$  by the same back-and-forth argument.)

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# Bibliography