

prf.1 Proof by Contradiction

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In the first instance, proof by contradiction is an inference pattern that is used to prove negative claims. Suppose you want to show that some claim p is *false*, i.e., you want to show $\neg p$. The most promising strategy is to (a) suppose that p is true, and (b) show that this assumption leads to something you know to be false. “Something known to be false” may be a result that conflicts with—contradicts— p itself, or some other hypothesis of the overall claim you are considering. For instance, a proof of “if q then $\neg p$ ” involves assuming that q is true and proving $\neg p$ from it. If you prove $\neg p$ by contradiction, that means assuming p in addition to q . If you can prove $\neg q$ from p , you have shown that the assumption p leads to something that contradicts your other assumption q , since q and $\neg q$ cannot both be true. Of course, you have to use other inference patterns in your proof of the contradiction, as well as unpacking definitions. Let’s consider an example.

Proposition prf.1. *If $A \subseteq B$ and $B = \emptyset$, then A has no *elements*.*

Proof. Suppose $A \subseteq B$ and $B = \emptyset$. We want to show that A has no *elements*.

Since this is a conditional claim, we assume the antecedent and want to prove the consequent. The consequent is: A has no *elements*. We can make that a bit more explicit: it’s not the case that there is an $x \in A$.

A has no *elements* iff it’s not the case that there is an x such that $x \in A$.

So we’ve determined that what we want to prove is really a negative claim $\neg p$, namely: it’s not the case that there is an $x \in A$. To use proof by contradiction, we have to assume the corresponding positive claim p , i.e., there is an $x \in A$, and prove a contradiction from it. We indicate that we’re doing a proof by contradiction by writing “by way of contradiction, assume” or even just “suppose not,” and then state the assumption p .

Suppose not: there is an $x \in A$.

This is now the new assumption we’ll use to obtain a contradiction. We have two more assumptions: that $A \subseteq B$ and that $B = \emptyset$. The first gives us that $x \in B$:

Since $A \subseteq B$, $x \in B$.

But since $B = \emptyset$, every *element* of B (e.g., x) must also be an *element* of \emptyset .

Since $B = \emptyset$, $x \in \emptyset$. This is a contradiction, since by definition \emptyset has no *elements*.

This already completes the proof: we've arrived at what we need (a contradiction) from the assumptions we've set up, and this means that the assumptions can't all be true. Since the first two assumptions ($A \subseteq B$ and $B = \emptyset$) are not contested, it must be the last assumption introduced (there is an $x \in A$) that must be false. But if we want to be thorough, we can spell this out.

Thus, our assumption that there is an $x \in A$ must be false, hence, A has no **elements** by proof by contradiction. \square

Every positive claim is trivially equivalent to a negative claim: p iff $\neg\neg p$. So proofs by contradiction can also be used to establish positive claims "indirectly," as follows: To prove p , read it as the negative claim $\neg\neg p$. If we can prove a contradiction from $\neg p$, we've established $\neg\neg p$ by proof by contradiction, and hence p .

In the last example, we aimed to prove a negative claim, namely that A has no **elements**, and so the assumption we made for the purpose of proof by contradiction (i.e., that there is an $x \in A$) was a positive claim. It gave us something to work with, namely the hypothetical $x \in A$ about which we continued to reason until we got to $x \in \emptyset$.

When proving a positive claim indirectly, the assumption you'd make for the purpose of proof by contradiction would be negative. But very often you can easily reformulate a positive claim as a negative claim, and a negative claim as a positive claim. Our previous proof would have been essentially the same had we proved " $A = \emptyset$ " instead of the negative consequent " A has no **elements**." (By definition of $=$, " $A = \emptyset$ " is a general claim, since it unpacks to "every **element** of A is an **element** of \emptyset and vice versa".) But it is easily seen to be equivalent to the negative claim "not: there is an $x \in A$."

So it is sometimes easier to work with $\neg p$ as an assumption than it is to prove p directly. Even when a direct proof is just as simple or even simpler (as in the next example), some people prefer to proceed indirectly. If the double negation confuses you, think of a proof by contradiction of some claim as a proof of a contradiction from the *opposite* claim. So, a proof by contradiction of $\neg p$ is a proof of a contradiction from the assumption p ; and proof by contradiction of p is a proof of a contradiction from $\neg p$.

Proposition prf.2. $A \subseteq A \cup B$.

Proof. We want to show that $A \subseteq A \cup B$.

On the face of it, this is a positive claim: every $x \in A$ is also in $A \cup B$. The negation of that is: some $x \in A$ is $\notin A \cup B$. So we can prove the claim indirectly by assuming this negated claim, and showing that it leads to a contradiction.

Suppose not, i.e., $A \not\subseteq A \cup B$.

We have a definition of $A \subseteq A \cup B$: every $x \in A$ is also $\in A \cup B$. To understand what $A \not\subseteq A \cup B$ means, we have to use some elementary logical manipulation on the unpacked definition: it's false that every $x \in A$ is also $\in A \cup B$ iff there is *some* $x \in A$ that is $\notin A \cup B$. (This is a place where you want to be very careful: many students' attempted proofs by contradiction fail because they analyze the negation of a claim like "all As are Bs" incorrectly.) In other words, $A \not\subseteq A \cup B$ iff there is an x such that $x \in A$ and $x \notin A \cup B$. From then on, it's easy.

So, there is an $x \in A$ such that $x \notin A \cup B$. By definition of \cup , $x \in A \cup B$ iff $x \in A$ or $x \in B$. Since $x \in A$, we have $x \in A \cup B$. This contradicts the assumption that $x \notin A \cup B$. \square

Problem prf.1. Prove *indirectly* that $A \cap B \subseteq A$.

Proposition prf.3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Proof. Suppose $A \subseteq B$ and $B \subseteq C$. We want to show $A \subseteq C$.

Let's proceed indirectly: we assume the negation of what we want to establish.

Suppose not, i.e., $A \not\subseteq C$.

As before, we reason that $A \not\subseteq C$ iff not every $x \in A$ is also $\in C$, i.e., some $x \in A$ is $\notin C$. Don't worry, with practice you won't have to think hard anymore to unpack negations like this.

In other words, there is an x such that $x \in A$ and $x \notin C$.

Now we can use this to get to our contradiction. Of course, we'll have to use the other two assumptions to do it.

Since $A \subseteq B$, $x \in B$. Since $B \subseteq C$, $x \in C$. But this contradicts $x \notin C$. \square

Proposition prf.4. If $A \cup B = A \cap B$ then $A = B$.

Proof. Suppose $A \cup B = A \cap B$. We want to show that $A = B$.

The beginning is now routine:

Assume, by way of contradiction, that $A \neq B$.

Our assumption for the proof by contradiction is that $A \neq B$. Since $A = B$ iff $A \subseteq B$ and $B \subseteq A$, we get that $A \neq B$ iff $A \not\subseteq B$ or $B \not\subseteq A$. (Note how important it is to be careful when manipulating negations!) To prove a contradiction from this disjunction, we use a proof by cases and show that in each case, a contradiction follows.

$A \neq B$ iff $A \not\subseteq B$ or $B \not\subseteq A$. We distinguish cases.

In the first case, we assume $A \not\subseteq B$, i.e., for some x , $x \in A$ but $x \notin B$. $A \cap B$ is defined as those **elements** that A and B have in common, so if something isn't in one of them, it's not in the intersection. $A \cup B$ is A together with B , so anything in either is also in the union. This tells us that $x \in A \cup B$ but $x \notin A \cap B$, and hence that $A \cap B \neq B \cap A$.

Case 1: $A \not\subseteq B$. Then for some x , $x \in A$ but $x \notin B$. Since $x \notin B$, then $x \notin A \cap B$. Since $x \in A$, $x \in A \cup B$. So, $A \cap B \neq B \cap A$, contradicting the assumption that $A \cap B = A \cup B$.

Case 2: $B \not\subseteq A$. Then for some y , $y \in B$ but $y \notin A$. As before, we have $y \in A \cup B$ but $y \notin A \cap B$, and so $A \cap B \neq A \cup B$, again contradicting $A \cap B = A \cup B$. \square

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Bibliography