

prf.1 Another Example

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sec

Proposition prf.1. *If $A \subseteq C$, then $A \cup (C \setminus A) = C$.*

Proof. Suppose that $A \subseteq C$. We want to show that $A \cup (C \setminus A) = C$.

We begin by observing that this is a conditional statement. It is tacitly universally quantified: the proposition holds for all sets A and C . So A and C are variables for arbitrary sets. To prove such a statement, we assume the antecedent and prove the consequent.

We continue by using the assumption that $A \subseteq C$. Let's unpack the definition of \subseteq : the assumption means that all **elements** of A are also **elements** of C . Let's write this down—it's an important fact that we'll use throughout the proof.

By the definition of \subseteq , since $A \subseteq C$, for all z , if $z \in A$, then $z \in C$.

We've unpacked all the definitions that are given to us in the assumption. Now we can move onto the conclusion. We want to show that $A \cup (C \setminus A) = C$, and so we set up a proof similarly to the last example: we show that every **element** of $A \cup (C \setminus A)$ is also **an element** of C and, conversely, every **element** of C is **an element** of $A \cup (C \setminus A)$. We can shorten this to: $A \cup (C \setminus A) \subseteq C$ and $C \subseteq A \cup (C \setminus A)$. (Here we're doing the opposite of unpacking a definition, but it makes the proof a bit easier to read.) Since this is a conjunction, we have to prove both parts. To show the first part, i.e., that every **element** of $A \cup (C \setminus A)$ is also **an element** of C , we assume that $z \in A \cup (C \setminus A)$ for an arbitrary z and show that $z \in C$. By the definition of \cup , we can conclude that $z \in A$ or $z \in C \setminus A$ from $z \in A \cup (C \setminus A)$. You should now be getting the hang of this.

$A \cup (C \setminus A) = C$ iff $A \cup (C \setminus A) \subseteq C$ and $C \subseteq (A \cup (C \setminus A))$. First we prove that $A \cup (C \setminus A) \subseteq C$. Let $z \in A \cup (C \setminus A)$. So, either $z \in A$ or $z \in (C \setminus A)$.

We've arrived at a disjunction, and from it we want to prove that $z \in C$. We do this using proof by cases.

Case 1: $z \in A$. Since for all z , if $z \in A$, $z \in C$, we have that $z \in C$.

Here we've used the fact recorded earlier which followed from the hypothesis of the proposition that $A \subseteq C$. The first case is complete, and we turn to the second case, $z \in (C \setminus A)$. Recall that $C \setminus A$ denotes the *difference* of the two sets, i.e., the set of all **elements** of C which are not **elements** of A . But any **element** of C not in A is in particular **an element** of C .

Case 2: $z \in (C \setminus A)$. This means that $z \in C$ and $z \notin A$. So, in particular, $z \in C$.

Great, we've proved the first direction. Now for the second direction. Here we prove that $C \subseteq A \cup (C \setminus A)$. So we assume that $z \in C$ and prove that $z \in A \cup (C \setminus A)$.

Now let $z \in C$. We want to show that $z \in A$ or $z \in C \setminus A$.

Since all **elements** of A are also **elements** of C , and $C \setminus A$ is the set of all things that are **elements** of C but not A , it follows that z is either in A or in $C \setminus A$. This may be a bit unclear if you don't already know why the result is true. It would be better to prove it step-by-step. It will help to use a simple fact which we can state without proof: $z \in A$ or $z \notin A$. This is called the "principle of excluded middle:" for any statement p , either p is true or its negation is true. (Here, p is the statement that $z \in A$.) Since this is a disjunction, we can again use proof-by-cases.

Either $z \in A$ or $z \notin A$. In the former case, $z \in A \cup (C \setminus A)$. In the latter case, $z \in C$ and $z \notin A$, so $z \in C \setminus A$. But then $z \in A \cup (C \setminus A)$.

Our proof is complete: we have shown that $A \cup (C \setminus A) = C$. □

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Bibliography