

## ind.1 Structural Induction

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sec So far we have used induction to establish results about all natural numbers. But a corresponding principle can be used directly to prove results about all **elements** of an inductively defined set. This is often called *structural* induction, because it depends on the structure of the inductively defined objects.

Generally, an inductive definition is given by (a) a list of “initial” **elements** of the set and (b) a list of operations which produce new **elements** of the set from old ones. In the case of nice terms, for instance, the initial objects are the letters. We only have one operation: the operations are

$$o(s_1, s_2) = [s_1 \circ s_2]$$

You can even think of the natural numbers  $\mathbb{N}$  themselves as being given by an inductive definition: the initial object is 0, and the operation is the successor function  $x + 1$ .

In order to prove something about all elements of an inductively defined set, i.e., that every **element** of the set has a property  $P$ , we must:

1. Prove that the initial objects have  $P$
2. Prove that for each operation  $o$ , if the arguments have  $P$ , so does the result.

For instance, in order to prove something about all nice terms, we would prove that it is true about all letters, and that it is true about  $[s_1 \circ s_2]$  provided it is true of  $s_1$  and  $s_2$  individually.

**Proposition ind.1.** *The number of [ equals the number of ] in any nice term  $t$ .*

*Proof.* We use structural induction. Nice terms are inductively defined, with letters as initial objects and the operation  $o$  for constructing new nice terms out of old ones.

1. The claim is true for every letter, since the number of [ in a letter by itself is 0 and the number of ] in it is also 0.
2. Suppose the number of [ in  $s_1$  equals the number of ], and the same is true for  $s_2$ . The number of [ in  $o(s_1, s_2)$ , i.e., in  $[s_1 \circ s_2]$ , is the sum of the number of [ in  $s_1$  and  $s_2$  plus one. The number of ] in  $o(s_1, s_2)$  is the sum of the number of ] in  $s_1$  and  $s_2$  plus one. Thus, the number of [ in  $o(s_1, s_2)$  equals the number of ] in  $o(s_1, s_2)$ .  $\square$

**Problem ind.1.** Prove by structural induction that no nice term starts with ] .

Let’s give another proof by structural induction: a proper initial segment of a string  $t$  of symbols is any string  $s$  that agrees with  $t$  symbol by symbol, read from the left, but  $t$  is longer. So, e.g.,  $[a \circ$  is a proper initial segment of  $[a \circ b]$ , but neither are  $[b \circ$  (they disagree at the second symbol) nor  $[a \circ b]$  (they are the same length).

**Proposition ind.2.** *Every proper initial segment of a nice term  $t$  has more [ 's than ]'s.* *math:ind:sti:  
prop:initial*

*Proof.* By induction on  $t$ :

1.  $t$  is a letter by itself: Then  $t$  has no proper initial segments.
2.  $t = [s_1 \circ s_2]$  for some nice terms  $s_1$  and  $s_2$ . If  $r$  is a proper initial segment of  $t$ , there are a number of possibilities:
  - a)  $r$  is just [: Then  $r$  has one more [ than it does ].
  - b)  $r$  is  $[r_1$  where  $r_1$  is a proper initial segment of  $s_1$ : Since  $s_1$  is a nice term, by induction hypothesis,  $r_1$  has more [ than ] and the same is true for  $[r_1$ .
  - c)  $r$  is  $[s_1$  or  $[s_1 \circ$ : By the previous result, the number of [ and ] in  $s_1$  are equal; so the number of [ in  $[s_1$  or  $[s_1 \circ$  is one more than the number of ].
  - d)  $r$  is  $[s_1 \circ r_2$  where  $r_2$  is a proper initial segment of  $s_2$ : By induction hypothesis,  $r_2$  contains more [ than ]. By the previous result, the number of [ and of ] in  $s_1$  are equal. So the number of [ in  $[s_1 \circ r_2$  is greater than the number of ].
  - e)  $r$  is  $[s_1 \circ s_2$ : By the previous result, the number of [ and ] in  $s_1$  are equal, and the same for  $s_2$ . So there is one more [ in  $[s_1 \circ s_2$  than there are ]. □

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