

ind.1 Structural Induction

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sec So far we have used induction to establish results about all natural numbers. But a corresponding principle can be used directly to prove results about all **elements** of an inductively defined set. This is often called *structural* induction, because it depends on the structure of the inductively defined objects.

Generally, an inductive definition is given by (a) a list of “initial” **elements** of the set and (b) a list of operations which produce new **elements** of the set from old ones. In the case of nice terms, for instance, the initial objects are the letters. We only have one operation: the operations are

$$o(s_1, s_2) = [s_1 \circ s_2]$$

You can even think of the natural numbers \mathbb{N} themselves as being given by an inductive definition: the initial object is 0, and the operation is the successor function $x + 1$.

In order to prove something about all elements of an inductively defined set, i.e., that every **element** of the set has a property P , we must:

1. Prove that the initial objects have P
2. Prove that for each operation o , if the arguments have P , so does the result.

For instance, in order to prove something about all nice terms, we would prove that it is true about all letters, and that it is true about $[s_1 \circ s_2]$ provided it is true of s_1 and s_2 individually.

Proposition ind.1. *The number of [equals the number of] in any nice term t .*

Proof. We use structural induction. Nice terms are inductively defined, with letters as initial objects and the operation o for constructing new nice terms out of old ones.

1. The claim is true for every letter, since the number of [in a letter by itself is 0 and the number of] in it is also 0.
2. Suppose the number of [in s_1 equals the number of], and the same is true for s_2 . The number of [in $o(s_1, s_2)$, i.e., in $[s_1 \circ s_2]$, is the sum of the number of [in s_1 and s_2 plus one. The number of] in $o(s_1, s_2)$ is the sum of the number of] in s_1 and s_2 plus one. Thus, the number of [in $o(s_1, s_2)$ equals the number of] in $o(s_1, s_2)$. \square

Problem ind.1. Prove by structural induction that no nice term starts with] .

Let’s give another proof by structural induction: a proper initial segment of a string t of symbols is any string s that agrees with t symbol by symbol, read from the left, but t is longer. So, e.g., $[a \circ$ is a proper initial segment of $[a \circ b]$, but neither are $[b \circ$ (they disagree at the second symbol) nor $[a \circ b]$ (they are the same length).

Proposition ind.2. *Every proper initial segment of a nice term t has more ['s than]'s.* *math:ind:sti:
prop:initial*

Proof. By induction on t :

1. t is a letter by itself: Then t has no proper initial segments.
2. $t = [s_1 \circ s_2]$ for some nice terms s_1 and s_2 . If r is a proper initial segment of t , there are a number of possibilities:
 - a) r is just [: Then r has one more [than it does].
 - b) r is $[r_1$ where r_1 is a proper initial segment of s_1 : Since s_1 is a nice term, by induction hypothesis, r_1 has more [than] and the same is true for $[r_1$.
 - c) r is $[s_1$ or $[s_1 \circ$: By the previous result, the number of [and] in s_1 are equal; so the number of [in $[s_1$ or $[s_1 \circ$ is one more than the number of].
 - d) r is $[s_1 \circ r_2$ where r_2 is a proper initial segment of s_2 : By induction hypothesis, r_2 contains more [than]. By the previous result, the number of [and of] in s_1 are equal. So the number of [in $[s_1 \circ r_2$ is greater than the number of].
 - e) r is $[s_1 \circ s_2$: By the previous result, the number of [and] in s_1 are equal, and the same for s_2 . So there is one more [in $[s_1 \circ s_2$ than there are]. □

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Bibliography