## ind.1 Strong Induction

mth:ind:str:sec In the principle of induction discussed above, we prove $P(0)$ and also if $P(n)$, then $P(n+1)$. In the second part, we assume that $P(n)$ is true and use this assumption to prove $P(n+1)$. Equivalently, of course, we could assume $P(n-1)$ and use it to prove $P(n)$—the important part is that we be able to carry out the inference from any number to its successor; that we can prove the claim in question for any number under the assumption it holds for its predecessor.

There is a variant of the principle of induction in which we don't just assume that the claim holds for the predecessor $n-1$ of $n$, but for all numbers smaller than $n$, and use this assumption to establish the claim for $n$. This also gives us the claim $P(k)$ for all $k \in \mathbb{N}$. For once we have established $P(0)$, we have thereby established that $P$ holds for all numbers less than 1. And if we know that if $P(l)$ for all $l < n$ then $P(n)$, we know this in particular for $n = 1$. So we can conclude $P(2)$. With this we have proved $P(0)$, $P(1)$, $P(2)$, i.e., $P(l)$ for all $l < 3$, and since we have also the conditional, if $P(l)$ for all $l < 3$, then $P(3)$, we can conclude $P(3)$, and so on.

In fact, if we can establish the general conditional "for all $n$, if $P(l)$ for all $l < n$, then $P(n)$," we do not have to establish $P(0)$ anymore, since it follows from it. For remember that a general claim like "for all $l < n$, $P(l)$" is true if there are no $l < n$. This is a case of vacuous quantification: "all $A$s are $B$s" is true if there are no $A$s, $\forall x\,(\varphi(x) \rightarrow \psi(x))$ is true if no $x$ satisfies $\varphi(x)$. In this case, the formalized version would be "$\forall l\,(l < n \rightarrow P(l))$"—and that is true if there are no $l < n$. And if $n = 0$ that's exactly the case: no $l < 0$, hence "for all $l < 0$, $P(0)$" is true, whatever $P$ is. A proof of "if $P(l)$ for all $l < n$, then $P(n)$" thus automatically establishes $P(0)$.

This variant is useful if establishing the claim for $n$ can't be made to just rely on the claim for $n-1$ but may require the assumption that it is true for one or more $l < n$.

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## Bibliography