ind.1 Relations and Functions

mth:ind:rel:sec When we have defined a set of objects (such as the natural numbers or the nice terms) inductively, we can also define *relations on* these objects by induction. For instance, consider the following idea: a nice term t_1 is a subterm of a nice term t_2 if it occurs as a part of it. Let's use a symbol for it: $t_1 \sqsubseteq t_2$. Every nice term is a subterm of itself, of course: $t \sqsubseteq t$. We can give an inductive definition of this relation as follows:

Definition ind.1. The relation of a nice term t_1 being a subterm of t_2, $t_1 \sqsubseteq t_2$, is defined by induction on t_2 as follows:

1. If t_2 is a letter, then $t_1 \sqsubseteq t_2$ iff $t_1 = t_2$.
2. If t_2 is $[s_1 \circ s_2]$, then $t_1 \sqsubseteq t_2$ iff $t_1 = t_2$, $t_1 \sqsubseteq s_1$, or $t_1 \sqsubseteq s_2$.

This definition, for instance, will tell us that $\mathrm{a} \sqsubseteq [\mathrm{b} \circ \mathrm{a}]$. For (2) says that $\mathrm{a} \sqsubseteq [\mathrm{b} \circ \mathrm{a}]$ iff $\mathrm{a} = [\mathrm{b} \circ \mathrm{a}]$, or $\mathrm{a} \sqsubseteq b$, or $\mathrm{a} \sqsubseteq \mathrm{a}$. The first two are false: a clearly isn't identical to $[\mathrm{b} \circ \mathrm{a}]$, and by (1), $\mathrm{a} \sqsubseteq \mathrm{b}$ iff $\mathrm{a} = \mathrm{b}$, which is also false. However, also by (1), $\mathrm{a} \sqsubseteq \mathrm{a}$ iff $\mathrm{a} = \mathrm{a}$, which is true.

It's important to note that the success of this definition depends on a fact that we haven't proved yet: every nice term t is either a letter by itself, or there are *uniquely determined* nice terms s_1 and s_2 such that $t = [s_1 \circ s_2]$. "Uniquely determined" here means that if $t = [s_1 \circ s_2]$ it isn't *also* $= [r_1 \circ r_2]$ with $s_1 \neq r_1$ or $s_2 \neq r_2$. If this were the case, then clause (2) may come in conflict with itself: reading t_2 as $[s_1 \circ s_2]$ we might get $t_1 \sqsubseteq t_2$, but if we read t_2 as $[r_1 \circ r_2]$ we might get not $t_1 \sqsubseteq t_2$. Before we prove that this can't happen, let's look at an example where it *can* happen.

Definition ind.2. Define *bracketless terms* inductively by

1. Every letter is a bracketless term.
2. If s_1 and s_2 are bracketless terms, then $s_1 \circ s_2$ is a bracketless term.
3. Nothing else is a bracketless term.

Bracketless terms are, e.g., a, $\mathrm{b} \circ \mathrm{d}$, $\mathrm{b} \circ \mathrm{a} \circ \mathrm{b}$. Now if we defined "subterm" for bracketless terms the way we did above, the second clause would read

If $t_2 = s_1 \circ s_2$, then $t_1 \sqsubseteq t_2$ iff $t_1 = t_2$, $t_1 \sqsubseteq s_1$, or $t_1 \sqsubseteq s_2$.

Now $\mathrm{b} \circ \mathrm{a} \circ \mathrm{b}$ is of the form $s_1 \circ s_2$ with

$$s_1 = \mathrm{b} \text{ and } \qquad s_2 = \mathrm{a} \circ \mathrm{b}.$$

It is also of the form $r_1 \circ r_2$ with

$$r_1 = \mathrm{b} \circ \mathrm{a} \text{ and } \qquad r_2 = \mathrm{b}.$$

Now is a ∘ b a subterm of b ∘ a ∘ b? The answer is yes if we go by the first reading, and no if we go by the second.

The property that the way a nice term is built up from other nice terms is unique is called *unique readability*. Since inductive definitions of relations for such inductively defined objects are important, we have to prove that it holds.

Proposition ind.3. *Suppose t is a nice term. Then either t is a letter by itself, or there are uniquely determined nice terms s_1, s_2 such that $t = [s_1 \circ s_2]$.*

Proof. If t is a letter by itself, the condition is satisfied. So assume t isn't a letter by itself. We can tell from the inductive definition that then t must be of the form $[s_1 \circ s_2]$ for some nice terms s_1 and s_2. It remains to show that these are uniquely determined, i.e., if $t = [r_1 \circ r_2]$, then $s_1 = r_1$ and $s_2 = r_2$.

So suppose $t = [s_1 \circ s_2]$ and also $t = [r_1 \circ r_2]$ for nice terms s_1, s_2, r_1, r_2. We have to show that $s_1 = r_1$ and $s_2 = r_2$. First, s_1 and r_1 must be identical, for otherwise one is a proper initial segment of the other. But by **??**, that is impossible if s_1 and r_1 are both nice terms. But if $s_1 = r_1$, then clearly also $s_2 = r_2$. □

We can also define functions inductively: e.g., we can define the function f that maps any nice term to the maximum depth of nested $[\dots]$ in it as follows:

Definition ind.4. The *depth* of a nice term, $f(t)$, is defined inductively as follows: mth:ind:rel: defn:depth

$$f(t) = \begin{cases} 0 & \text{if } t \text{ is a letter} \\ \max(f(s_1), f(s_2)) + 1 & \text{if } t = [s_1 \circ s_2]. \end{cases}$$

For instance

$$\begin{aligned} f([\mathrm{a} \circ \mathrm{b}]) &= \max(f(\mathrm{a}), f(\mathrm{b})) + 1 = \\ &= \max(0, 0) + 1 = 1, \text{ and} \\ f([[\mathrm{a} \circ \mathrm{b}] \circ \mathrm{c}]) &= \max(f([\mathrm{a} \circ \mathrm{b}]), f(\mathrm{c})) + 1 = \\ &= \max(1, 0) + 1 = 2. \end{aligned}$$

Here, of course, we assume that s_1 an s_2 are nice terms, and make use of the fact that every nice term is either a letter or of the form $[s_1 \circ s_2]$. It is again important that it can be of this form in only one way. To see why, consider again the bracketless terms we defined earlier. The corresponding "definition" would be:

$$g(t) = \begin{cases} 0 & \text{if } t \text{ is a letter} \\ \max(g(s_1), g(s_2)) + 1 & \text{if } t = s_1 \circ s_2. \end{cases}$$

Now consider the bracketless term a ∘ b ∘ c ∘ d. It can be read in more than one way, e.g., as $s_1 \circ s_2$ with

$$s_1 = \mathrm{a} \text{ and} \qquad s_2 = \mathrm{b} \circ \mathrm{c} \circ \mathrm{d},$$

or as $r_1 \circ r_2$ with

$$r_1 = \mathrm{a} \circ b \text{ and} \qquad\qquad r_2 = \mathrm{c} \circ \mathrm{d}.$$

Calculating g according to the first way of reading it would give

$$\begin{aligned} g(s_1 \circ s_2) &= \max(g(\mathrm{a}), g(\mathrm{b} \circ \mathrm{c} \circ \mathrm{d})) + 1 = \\ &= \max(0, 2) + 1 = 3 \end{aligned}$$

while according to the other reading we get

$$\begin{aligned} g(r_1 \circ r_2) &= \max(g(\mathrm{a} \circ \mathrm{b}), g(\mathrm{c} \circ \mathrm{d})) + 1 = \\ &= \max(1, 1) + 1 = 2 \end{aligned}$$

But a function must always yield a unique value; so our "definition" of g doesn't define a function at all.

Problem ind.1. Give an inductive definition of the function l, where $l(t)$ is the number of symbols in the nice term t.

Problem ind.2. Prove by structural induction on nice terms t that $f(t) < l(t)$ (where $l(t)$ is the number of symbols in t and $f(t)$ is the depth of t as defined in Definition ind.4).

Photo Credits

Bibliography