Chapter udf

Three-valued Logics

thr.1 Introduction

mvl:thr:int:sec If we just add one more value $\mathbb{U}$ to $\mathbb{T}$ and $\mathbb{F}$, we get a three-valued logic. Even though there is only one more truth value, the possibilities for defining the truth-functions for $\neg$, $\wedge$, $\vee$, and $\rightarrow$ are quite numerous. Then a logic might use any combination of these truth functions, and you also have a choice of making only $\mathbb{T}$ designated, or both $\mathbb{T}$ and $\mathbb{U}$.

We present here a selection of the most well-known three-valued logics, their motivations, and some of their properties.

thr.2 Łukasiewicz logic

mvl:thr:luk:sec One of the first published, worked out proposals for a many-valued logic is due to the Polish philosopher Jan Łukasiewicz in 1921. Łukasiewicz was motivated by Aristotle's sea battle problem: It seems that, *today*, the sentence "There will be a sea battle tomorrow" is neither true nor false: its truth value is not yet settled. Łukasiewicz proposed to introduce a third truth value, to such "future contingent" sentences.

> I can assume without contradiction that my presence in Warsaw at a certain moment of next year, e.g., at noon on 21 December, is at the present time determined neither positively nor negatively. Hence it is possible, but not necessary, that I shall be present in Warsaw at the given time. On this assumption the proposition "I shall be in Warsaw at noon on 21 December of next year," can at the present time be neither true nor false. For if it were true now, my future presence in Warsaw would have to be necessary, which is contradictory to the assumption. If it were false now, on the other hand, my future presence in Warsaw would have to be impossible, which is also contradictory to the assumption. Therefore the proposition considered is at the moment neither true nor false and must possess a third value, different from "0" or falsity and "1" or truth.

> This value we can designate by "$\frac{1}{2}$." It represents "the possible," and joins "the true" and "the false" as a third value.

We will use $\mathbb{U}$ for Łukasiewicz's third truth value.[1]

The truth functions for the connectives $\neg$, $\wedge$, and $\vee$ are easy to determine on this interpretation: the negation of a future contingent sentence is also a future contingent sentence, so $\widetilde{\neg}(\mathbb{U}) = \mathbb{U}$. If one conjunct of a conjunction is undetermined and the other is true, the conjunction is also undetermined—after all, depending on how the future contingent conjunct turns out, the conjunction might turn out to be true, and it might turn out to be false. So

$$\widetilde{\wedge}(\mathbb{T}, \mathbb{U}) = \widetilde{\wedge}(\mathbb{U}, \mathbb{T}) = \mathbb{U}.$$

If the other conjunct is false, however, it cannot turn out true, so

$$\widetilde{\wedge}(\mathbb{F}, \mathbb{U}) = \widetilde{\wedge}(\mathbb{F}, \mathbb{U}) = \mathbb{F}.$$

The other values (if the arguments are settled truth values, $\mathbb{T}$ or $\mathbb{F}$, are like in classical logic.

For the conditional, the situation is a little trickier. Suppose q is a future contingent statement. If p is false, then $p \to q$ will be true, regardless of how q turns out, so we should set $\widetilde{\to}(\mathbb{F}, \mathbb{U}) = \mathbb{T}$. And if p is true, then $q \to p$ will be true, regardless of what q turns out to be, so $\widetilde{\to}(\mathbb{U}, \mathbb{T}) = \mathbb{T}$. If p is true, then $p \to q$ might turn out to be true or false, so $\widetilde{\to}(\mathbb{T}, \mathbb{U}) = \mathbb{U}$. Similarly, if p is false, then $q \to p$ might turn out to be true or false, so $\widetilde{\to}(\mathbb{U}, \mathbb{F}) = \mathbb{U}$. This leaves the case where p and q are both future contingents. On the basis of the motivation, we should really assign $\mathbb{U}$ in this case. However, this would make $\varphi \to \varphi$ *not* a tautology. Łukasiewicz had not trouble giving up $\varphi \vee \neg\varphi$ and $\neg(\varphi \wedge \neg\varphi)$, but balked at giving up $\varphi \to \varphi$. So he stipulated $\widetilde{\to}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

Definition thr.1. Three-valued Łukasiewicz logic is defined using the matrix: mvl:thr:luk: def:lukasiewicz

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\wedge$, $\vee$, $\to$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

$\widetilde{\neg}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{\wedge}_{\mathbf{Ł}_3}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$\widetilde{\vee}_{\mathbf{Ł}_3}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\to}_{\mathbf{Ł}_3}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

[1] Łukasiewicz here uses "possible" in a way that is uncommon today, namely to mean possible but not necessary.

As can easily be seen, any formula φ containing only $\neg$, $\wedge$, and $\vee$ will take the truth value $\mathbb{U}$ if all its propositional variables are assigned $\mathbb{U}$. So for instance, the classical tautologies $p \vee \neg p$ and $\neg(p \wedge \neg p)$ are not tautologies in $\mathbf{Ł}_3$, since $\overline{\mathfrak{v}}(\varphi) = \mathbb{U}$ whenever $\mathfrak{v}(p) = \mathbb{U}$.

On valuations where $\mathfrak{v}(p) = \mathbb{T}$ or $\mathbb{F}$, $\overline{\mathfrak{v}}(\varphi)$ will coincide with its classical truth value.

Proposition thr.2. *If $\mathfrak{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$ for all p in φ, then $\overline{\mathfrak{v}}_{\mathbf{Ł}_3}(\varphi) = \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi)$.*

Problem thr.1. Suppose we define $\overline{\mathfrak{v}}(\varphi \leftrightarrow \psi) = \overline{\mathfrak{v}}((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ in $\mathbf{Ł}_3$. What truth table would $\leftrightarrow$ have?

Many classical tautologies *are* also tautologies in $\mathbf{Ł}_3$, e.g, $\neg p \rightarrow (p \rightarrow q)$. Just like in classical logic, we can use truth tables to verify this:

p	q	$\neg$	p	$\rightarrow$	$(p$	$\rightarrow$	$q)$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$
$\mathbb{U}$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{U}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

Problem thr.2. Show that the following are tautologies in $\mathbf{Ł}_3$:

1. $p \rightarrow (q \rightarrow p)$
2. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
3. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

(In (2) and (3), take $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, or refer to your solution to Problem thr.1.)

Problem thr.3. Show that the following classical tautologies are not tautologies in $\mathbf{Ł}_3$:

1. $(\neg p \wedge p) \rightarrow q)$
2. $((p \rightarrow q) \rightarrow p) \rightarrow p$
3. $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$

One might therefore perhaps think that although not all classical tautologies are tautologies in $\mathbf{Ł}_3$, they should at least take either the value $\mathbb{T}$ or the value $\mathbb{U}$ on every valuation. This is not the case. A counterexample is given by

$$\neg(p \rightarrow \neg p) \vee \neg(\neg p \rightarrow p)$$

which is $\mathbb{F}$ if p is $\mathbb{U}$.

Problem thr.4. Which of the following relations hold in Łukasiewicz logic? Give a truth table for each.

1. $p, p \rightarrow q \vDash q$
2. $\neg\neg p \vDash p$
3. $p \wedge q \vDash p$
4. $p \vDash p \wedge p$
5. $p \vDash p \vee q$

Łukasiewicz hoped to build a logic of possibility on the basis of his three-valued system, by introducing a one-place connective $\Diamond\varphi$ (for "φ is possible") and a corresponding $\Box\varphi$ (for "φ is necessary"):

$$\begin{array}{c|c} \widetilde{\Diamond} & \\ \hline \mathbb{T} & \mathbb{T} \\ \mathbb{U} & \mathbb{T} \\ \mathbb{F} & \mathbb{F} \end{array} \qquad \begin{array}{c|c} \widetilde{\Box} & \\ \hline \mathbb{T} & \mathbb{T} \\ \mathbb{U} & \mathbb{F} \\ \mathbb{F} & \mathbb{F} \end{array}$$

In other words, p is possible iff it is not already settled as false; and p is necessary iff it is already settled as true.

Problem thr.5. Show that $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ are tautologies in $\mathbf{Ł}_3$, extended with the truth tables for $\Box$ and $\Diamond$.

However, the shortcomings of this proposed modal logic soon became evident: However things turn out, $p \wedge \neg p$ can never turn out to be true. So even if it is not now settled (and therefore undetermined), it should count as impossible, i.e., $\neg\Diamond(p \wedge \neg p)$ should be a tautology. However, if $\mathfrak{v}(p) = \mathbb{U}$, then $\overline{\mathfrak{v}}(\neg\Diamond(p \wedge \neg p)) = \mathbb{U}$. Although Łukasiewicz was correct that two truth values will not be enough to accommodate modal distinctions such as possiblity and necessity, introducing a third truth value is also not enough.

thr.3 Kleene logics

Stephen Kleene introduced two three-valued logics motivated by a logic in which truth values are thought of the outcomes of computational procedures: a procedure may yield $\mathbb{T}$ or $\mathbb{F}$, but it may also fail to terminate. In that case the corresponding truth value is undefined, represented by the truth value $\mathbb{U}$. mvl:thr:skl:sec

To compute the negation of a proposition φ, you would first compute the value of φ, and then return the opposite of the result. If the computation of φ does not terminate, then the entire procedure does not either: so the negation of $\mathbb{U}$ is $\mathbb{U}$.

To compute a conjunction $\varphi \wedge \psi$, there are two options: one can first compute φ, then ψ, and then the result would be $\mathbb{T}$ if the outcome of both is $\mathbb{T}$, and $\mathbb{F}$ otherwise. If either computation fails to halt, the entire procedure does as well. So in this case, the if one conjunct is undefined, the conjunction is as well. The same goes for disjunction.

However, if we can evaluate φ and ψ in parallel, we can do better. Then, if one of the two procedures halts and returns $\mathbb{F}$, we can stop, as the answer must be false. So in that case a conjunction with one false conjunct is false, even if the other conjunct is undefined. Similarly, when computing a disjunction in parallel, we can stop once the procedure for one of the two disjuncts has returned true: then the disjunction must be true. So in this case we can know what the outcome of a compound claim is, even if one of the components is undefined. On this interpretation, we might read $\mathbb{U}$ as "unknown" rather than "undefined."

The two interpretations give rise to Kleene's strong and weak logic. The conditional is defined as equivalent to $\neg\varphi \lor \psi$.

Definition thr.3. *Strong Kleene logic* **Ks** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

$\widetilde{\neg}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{\land}_{\mathbf{Ks}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$\widetilde{\lor}_{\mathbf{Ks}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\rightarrow}_{\mathbf{Ks}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

Definition thr.4. *Weak Kleene logic* **Kw** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

$\widetilde{\neg}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{\land}_{\mathbf{Kw}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\lor}_{\mathbf{Kw}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\to}_{\mathbf{Kw}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{T}$

Proposition thr.5. **Ks** *and* **Kw** *have no tautologies.*

Proof. If $\mathfrak{v}(p) = \mathbb{U}$ for all propositional variables p, then any formula φ will have truth value $\overline{\mathfrak{v}}(\varphi) = \mathbb{U}$, since

$$\widetilde{\neg}(\mathbb{U}) = \widetilde{\lor}(\mathbb{U}, \mathbb{U}) = \widetilde{\land}(\mathbb{U}, \mathbb{U}) = \widetilde{\to}(\mathbb{U}, \mathbb{U}) = \mathbb{U}$$

in both logics. As $\mathbb{U} \notin V^+$ for either **Ks** or **Kw**, on this valuation, φ will not be designated. □

Although both weak and strong Kleene logic have no tautologies, they have non-trivial consequence relations.

Problem thr.6. Which of the following relations hold in (a) strong and (b) weak Kleene logic? Give a truth table for each.

1. $p, p \to q \vDash q$
2. $p \lor q, \neg p \vDash q$
3. $p \land q \vDash p$
4. $p \vDash p \land p$
5. $p \vDash p \lor q$

Dmitry Bochvar interpreted $\mathbb{U}$ as "meaningless" and attempted to use it to solve paradoxes such as the Liar paradox by stipulating that paradoxical sentences take the value $\mathbb{U}$. He introduced a logic which is essentially weak Kleene logic extended by additional connectives, two of which are "external negation" and the "is undefined" operator:

$\widetilde{\sim}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{+}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$

Problem thr.7. Can you define $\sim$ in Bochvar's logic in terms of $\neg$ and $+$, i.e., find a formula with only the propositional variable p and not involving $\sim$ which always takes the same truth value as $\sim p$? Give a truth table to show you're right.

thr.4 Gödel logics

mvl:thr:god:sec Kurt Gödel introduced a sequence of n-valued logics that each contain all formulas valid in intuitionistic logic, and are contained in classical logic. Here is the first interesting one:

mvl:thr:god:defn:goedel **Definition thr.6.** 3-*valued Gödel logic* **G** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\wedge$, $\vee$, $\to$.

2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.

3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.

4. For $\bot$, we have $\widetilde{\bot} = \mathbb{F}$. Truth functions for the remaining connectives are given by the following tables:

$\widetilde{\neg}_{\mathbf{G}}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{\wedge}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$\widetilde{\vee}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\to}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

You'll notice that the truth tables for $\wedge$ and $\vee$ are the same as in Łukasiewicz and strong Kleene logic, but the truth tables for $\neg$ and $\to$ differ for each. In Gödel logic, $\widetilde{\neg}(\mathbb{U}) = \mathbb{F}$. In contrast to Łukasiewicz logic and Kleene logic, $\widetilde{\to}(\mathbb{U}, \mathbb{F}) = \mathbb{F}$; in contrast to Kleene logic (but as in Łukasiewicz logic), $\widetilde{\to}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

As the connection to intuitionistic logic alluded to above suggests, $\mathbf{G}_3$ is close to intuitionistic logic. All intuitionistic truths are tautologies in $\mathbf{G}_3$, and many classical tautologies that are not valid intuitionistically also fail to be tautologies in $\mathbf{G}_3$. For instance, the following are not tautologies:

$$
\begin{array}{ll}
p \vee \neg p & (p \to q) \to (\neg p \vee q) \\
\neg\neg p \to p & \neg(p \wedge q) \to (\neg p \vee \neg q) \\
 & ((p \to q) \to p) \to p
\end{array}
$$

However, not every tautology of $\mathbf{G}_3$ is also intuitionistically valid, e.g., $(p \to q) \vee (q \to p)$.

Problem thr.8. Give a truth table to show that $(p \to q) \vee (q \to p)$ is a tautology of $\mathbf{G}_3$.

Problem thr.9. Give truth tables that show that the following are not tautologies of $\mathbf{G}_3$

$$(p \to q) \to (\neg p \lor q)$$
$$\neg(p \land q) \to (\neg p \lor \neg q)$$
$$((p \to q) \to p) \to p$$

Problem thr.10. Which of the following relations hold in Gödel logic? Give a truth table for each.

1. $p, p \to q \vDash q$
2. $p \lor q, \neg p \vDash q$
3. $p \land q \vDash p$
4. $p \vDash p \land p$
5. $p \vDash p \lor q$

thr.5 Designating not just $\mathbb{T}$

So far the logics we've seen all had the set of designated truth values $V^+ = \{\mathbb{T}\}$, i.e., something counts as true iff its truth value is $\mathbb{T}$. But one might also count something as true if it's just not $\mathbb{F}$. Then one would get a logic by stipulating in the matrix, e.g., that $V^+ = \{\mathbb{T}, \mathbb{U}\}$. mvl:thr:mul:sec

Definition thr.7. The *logic of paradox* **LP** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\to$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as in strong Kleene logic.

Definition thr.8. Halldén's *logic of nonsense* **Hal** is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\to$ and a 1-place connective $+$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as weak Kleene logic, plus the "is meaningless" operator:

$\widetilde{+}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$

By contrast to the Kleene logics with which they share truth tables, these *do* have tautologies.

mvl:thr:mul: prop:LP-taut-CL **Proposition thr.9.** *The tautologies of* **LP** *are the same as the tautologies of classical propositional logic.*

Proof. By **??**, if $\vDash_{\mathbf{LP}} \varphi$ then $\vDash_{\mathbf{C}} \varphi$. To show the reverse, we show that if there is a valuation $\mathfrak{v}\colon \mathrm{At}_0 \to \{\mathbb{F}, \mathbb{T}, \mathbb{U}\}$ such that $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then there is a valuation $\mathfrak{v}'\colon \mathrm{At}_0 \to \{\mathbb{F}, \mathbb{T}\}$ such that $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$. This establishes the result for **LP**, since **Ks** and **LP** have the same characteristic truth functions, and $\mathbb{F}$ is the only truth value of **LP** that is not designated (that is the only difference between **LP** and **Ks**). Thus, if $\nvDash_{\mathbf{LP}} \varphi$, for some valuation $\mathfrak{v}$, $\overline{\mathfrak{v}}_{\mathbf{LP}}(\varphi) = \overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$. By the claim we're proving, $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$, i.e., $\nvDash_{\mathbf{C}} \varphi$.

To establish the claim, we first define $\mathfrak{v}'$ as

$$\mathfrak{v}'(p) = \begin{cases} \mathbb{T} & \text{if } \mathfrak{v}(p) \in \{\mathbb{T}, \mathbb{U}\} \\ \mathbb{F} & \text{otherwise} \end{cases}$$

We now show by induction on φ that (a) if $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$, and (b) if $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{T}$ then $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{T}$

1. Induction basis: $\varphi \equiv p$. By **??**, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\varphi) = \mathfrak{v}(p) = \overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi)$, which implies both (a) and (b).

   For the induction step, consider the cases:

2. $\varphi \equiv \neg\psi$.

   a) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{F}$. By the definition of $\widetilde{\neg}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{T}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\neg\psi) = \mathbb{F}$.

   b) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{T}$. By the definition of $\widetilde{\neg}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{F}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\neg\psi) = \mathbb{T}$.

3. $\varphi \equiv (\psi \wedge \chi)$.

   a) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{F}$. By the definition of $\widetilde{\wedge}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$ or $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{F}$ or $\overline{\mathfrak{v}'}_{\mathbf{C}}(\chi) = \mathbb{F}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{F}$.

   b) Suppose $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{T}$. By the definition of $\widetilde{\wedge}_{\mathbf{Ks}}$, $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$ and $\overline{\mathfrak{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi) = \mathbb{T}$ and $\overline{\mathfrak{v}'}_{\mathbf{C}}(\chi) = \mathbb{T}$, so $\overline{\mathfrak{v}'}_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{T}$.

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all formulas only containing $\neg$ and $\land$. One may now appeal to the facts that in both **Ks** and **C**, for any $\mathfrak{v}$, $\overline{\mathfrak{v}}(\psi \lor \chi) = \overline{\mathfrak{v}}(\neg(\neg\psi \land \neg\chi))$ and $\overline{\mathfrak{v}}(\psi \to \chi) = \overline{\mathfrak{v}}(\neg(\psi \land \neg\chi))$. □

Problem thr.11. Complete the proof Proposition thr.9, i.e., establish (a) and (b) for the cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \to \chi)$.

Problem thr.12. Prove that every classical tautology is a tautology in **Hal**.

Although they have the same tautologies as classical logic, their consequence relations are different. **LP**, for instance, is *paraconsistent* in that $\neg p, p \nvDash q$, and so the principle of explosion $\neg\varphi, \varphi \vDash \psi$ does not hold in general. (It holds for some cases of φ and ψ, e.g., if ψ is a tautology.)

Problem thr.13. Which of the following relations hold in (a) **LP** and in (b) **Hal**? Give a truth table for each.

1. $p, p \to q \vDash q$
2. $\neg q, p \to q \vDash \neg p$
3. $p \lor q, \neg p \vDash q$
4. $\neg p, p \vDash q$
5. $p \vDash p \lor q$
6. $p \to q, q \to r \vDash p \to r$

What if you make $\mathbb{U}$ designated in $\mathbf{Ł}_3$?

Definition thr.10. The logic *3-valued R-Mingle* $\mathbf{RM}_3$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\land$, $\lor$, $\to$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. $\mathbb{T}$ and $\mathbb{U}$ are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as Łukasiewicz logic $\mathbf{Ł}_3$.

Problem thr.14. Which of the following relations hold in $\mathbf{RM}_3$?

1. $p, p \to q \vDash q$
2. $p \lor q, \neg p \vDash q$
3. $\neg p, p \vDash q$
4. $p \vDash p \lor q$

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take $V^+ = \{\mathbb{T}, \mathbb{U}\}$ instead of $\{\mathbb{T}\}$.

mvl:thr:mul: prop:gl-udes **Proposition thr.11.** *The matrix with* $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$*,* $V^+ = \{\mathbb{T}, \mathbb{U}\}$*, and the truth functions of* 3*-valued Gödel logic defines classical logic.*

Proof. Exercise. □

Problem thr.15. Prove Proposition thr.11 by showing that for the logic $\mathbf{L}$ defined just like Gödel logic but with $V^+ = \{\mathbb{T}, \mathbb{U}\}$, if $\Gamma \nvDash_{\mathbf{L}} \psi$ then $\Gamma \nvDash_{\mathbf{C}} \psi$. Use the ideas of Proposition thr.9, except instead of proving properties (a) and (b), show that $\overline{\mathfrak{v}}_{\mathbf{G}}(\varphi) = \mathbb{F}$ iff $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{F}$ (and hence that $\overline{\mathfrak{v}}_{\mathbf{G}}(\varphi) \in \{\mathbb{T}, \mathbb{U}\}$ iff $\overline{\mathfrak{v}'}_{\mathbf{C}}(\varphi) = \mathbb{T}$). Explain why this establishes the proposition.

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Bibliography