

thr.1 Designating not just \mathbb{T}

mvl:thr:mul:
sec

So far the logics we've seen all had the set of designated truth values $V^+ = \{\mathbb{T}\}$, i.e., something counts as true iff its truth value is \mathbb{T} . But one might also count something as true if it's just not \mathbb{F} . Then one would get a logic by stipulating in the matrix, e.g., that $V^+ = \{\mathbb{T}, \mathbb{U}\}$.

Definition thr.1. The *logic of paradox LP* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as in strong Kleene logic.

Definition thr.2. Halldén's *logic of nonsense Hal* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$ and a 1-place connective $+$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as weak Kleene logic, plus the “is meaningless” operator:

$\tilde{+}$	\mathbb{F}
\mathbb{T}	\mathbb{F}
\mathbb{U}	\mathbb{T}
\mathbb{F}	\mathbb{F}

By contrast to the Kleene logics with which they share truth tables, these *do* have tautologies.

mul:thr:mul:
prop:LP-taut-CL

Proposition thr.3. *The tautologies of LP are the same as the tautologies of classical propositional logic.*

Proof. By ??, if $\models_{\mathbf{LP}} \varphi$ then $\models_{\mathbf{C}} \varphi$. To show the reverse, we show that if there is a **valuation** $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{F}, \mathbb{T}, \mathbb{U}\}$ such that $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then there is a **valuation** $\mathbf{v}': \text{At}_0 \rightarrow \{\mathbb{F}, \mathbb{T}\}$ such that $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$. This establishes the result for **LP**, since **Ks** and **LP** have the same characteristic truth functions, and \mathbb{F} is the only truth value of **LP** that is not designated (that is the only difference between **LP** and **Ks**). Thus, if $\not\models_{\mathbf{LP}} \varphi$, for some **valuation** \mathbf{v} , $\bar{\mathbf{v}}_{\mathbf{LP}}(\varphi) = \bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$. By the claim we're proving, $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$, i.e., $\not\models_{\mathbf{C}} \varphi$.

To establish the claim, we first define \mathbf{v}' as

$$\mathbf{v}'(p) = \begin{cases} \mathbb{T} & \text{if } \mathbf{v}(p) \in \{\mathbb{T}, \mathbb{U}\} \\ \mathbb{F} & \text{otherwise} \end{cases}$$

We now show by induction on φ that (a) if $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{F}$ then $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$, and (b) if $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbb{T}$ then $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{T}$

1. Induction basis: $\varphi \equiv p$. By ??, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\varphi) = \mathbf{v}(p) = \bar{\mathbf{v}}'_{\mathbf{C}}(\varphi)$, which implies both (a) and (b).

For the induction step, consider the cases:

2. $\varphi \equiv \neg\psi$.
 - a) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{F}$. By the definition of $\tilde{\bar{\mathbf{v}}}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{T}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\neg\psi) = \mathbb{F}$.
 - b) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\neg\psi) = \mathbb{T}$. By the definition of $\tilde{\bar{\mathbf{v}}}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{F}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\neg\psi) = \mathbb{T}$.
3. $\varphi \equiv (\psi \wedge \chi)$.
 - a) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{F}$. By the definition of $\tilde{\bar{\mathbf{v}}}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{F}$ or $\bar{\mathbf{v}}_{\mathbf{Ks}}(\chi) = \mathbb{F}$. By inductive hypothesis, case (a), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{F}$ or $\bar{\mathbf{v}}'_{\mathbf{C}}(\chi) = \mathbb{F}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{F}$.
 - b) Suppose $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi \wedge \chi) = \mathbb{T}$. By the definition of $\tilde{\bar{\mathbf{v}}}_{\mathbf{Ks}}$, $\bar{\mathbf{v}}_{\mathbf{Ks}}(\psi) = \mathbb{T}$ and $\bar{\mathbf{v}}_{\mathbf{Ks}}(\chi) = \mathbb{T}$. By inductive hypothesis, case (b), we get $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi) = \mathbb{T}$ and $\bar{\mathbf{v}}'_{\mathbf{C}}(\chi) = \mathbb{T}$, so $\bar{\mathbf{v}}'_{\mathbf{C}}(\psi \wedge \chi) = \mathbb{T}$.

The other two cases are similar, and left as exercises. Alternatively, the proof above establishes the result for all **formulas** only containing \neg and \wedge . One may now appeal to the facts that in both \mathbf{Ks} and \mathbf{C} , for any \mathbf{v} , $\bar{\mathbf{v}}(\psi \vee \chi) = \bar{\mathbf{v}}(\neg(\neg\psi \wedge \neg\chi))$ and $\bar{\mathbf{v}}(\psi \rightarrow \chi) = \bar{\mathbf{v}}(\neg(\psi \wedge \neg\chi))$. \square

Problem thr.1. Complete the proof **Proposition thr.3**, i.e., establish (a) and (b) for the cases where $\varphi \equiv (\psi \vee \chi)$ and $\varphi \equiv (\psi \rightarrow \chi)$.

Problem thr.2. Prove that every classical tautology is a tautology in **Hal**.

Although they have the same tautologies as classical logic, their consequence relations are different. **LP**, for instance, is *paraconsistent* in that $\neg p, p \not\vdash q$, and so the principle of explosion $\neg\varphi, \varphi \vdash \psi$ does not hold in general. (It holds for some cases of φ and ψ , e.g., if ψ is a tautology.)

Problem thr.3. Which of the following relations hold in (a) **LP** and in (b) **Hal**? Give a truth table for each.

1. $p, p \rightarrow q \vdash q$
2. $\neg q, p \rightarrow q \vdash \neg p$
3. $p \vee q, \neg p \vdash q$
4. $\neg p, p \vdash q$
5. $p \vdash p \vee q$

6. $p \rightarrow q, q \rightarrow r \models p \rightarrow r$

What if you make \mathbb{U} designated in \mathbf{L}_3 ?

Definition thr.4. The logic *3-valued R-Mingle* \mathbf{RM}_3 is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} and \mathbb{U} are designated, i.e., $V^+ = \{\mathbb{T}, \mathbb{U}\}$.
4. Truth functions are the same as Łukasiewicz logic \mathbf{L}_3 .

Problem thr.4. Which of the following relations hold in \mathbf{RM}_3 ?

1. $p, p \rightarrow q \models q$
2. $p \vee q, \neg p \models q$
3. $\neg p, p \models q$
4. $p \models p \vee q$

Different truth tables can sometimes generate the same logic (entailment relation) just by changing the designated values. E.g., this happens if in Gödel logic we take $V^+ = \{\mathbb{T}, \mathbb{U}\}$ instead of $\{\mathbb{T}\}$.

mul:thr:mul: **Proposition thr.5.** *prop:gl-udes* The matrix with $V = \{\mathbb{F}, \mathbb{U}, \mathbb{T}\}$, $V^+ = \{\mathbb{T}, \mathbb{U}\}$, and the truth functions of 3-valued Gödel logic defines classical logic.

Proof. Exercise. □

Problem thr.5. Prove **Proposition thr.5** by showing that for the logic \mathbf{L} defined just like Gödel logic but with $V^+ = \{\mathbb{T}, \mathbb{U}\}$, if $\Gamma \not\models_{\mathbf{L}} \psi$ then $\Gamma \not\models_{\mathbf{C}} \psi$. Use the ideas of **Proposition thr.3**, except instead of proving properties (a) and (b), show that $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) = \mathbb{F}$ iff $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{F}$ (and hence that $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) \in \{\mathbb{T}, \mathbb{U}\}$ iff $\bar{\mathbf{v}}'_{\mathbf{C}}(\varphi) = \mathbb{T}$). Explain why this establishes the proposition.

Photo Credits

Bibliography