## thr.1 Łukasiewicz logic

mvl:thr:luk:sec One of the first published, worked out proposals for a many-valued logic is due to the Polish philosopher Jan Łukasiewicz in 1921. Łukasiewicz was motivated by Aristotle's sea battle problem: It seems that, *today*, the sentence "There will be a sea battle tomorrow" is neither true nor false: its truth value is not yet settled. Łukasiewicz proposed to introduce a third truth value, to such "future contingent" sentences.

> I can assume without contradiction that my presence in Warsaw at a certain moment of next year, e.g., at noon on 21 December, is at the present time determined neither positively nor negatively. Hence it is possible, but not necessary, that I shall be present in Warsaw at the given time. On this assumption the proposition "I shall be in Warsaw at noon on 21 December of next year," can at the present time be neither true nor false. For if it were true now, my future presence in Warsaw would have to be necessary, which is contradictory to the assumption. If it were false now, on the other hand, my future presence in Warsaw would have to be impossible, which is also contradictory to the assumption. Therefore the proposition considered is at the moment neither true nor false and must possess a third value, different from "0" or falsity and "1" or truth. This value we can designate by "$\frac{1}{2}$." It represents "the possible," and joins "the true" and "the false" as a third value.

We will use $\mathbb{U}$ for Łukasiewicz's third truth value.[1]

The truth functions for the connectives $\neg$, $\land$, and $\lor$ are easy to determine on this interpretation: the negation of a future contingent sentence is also a future contingent sentence, so $\widetilde{\neg}(\mathbb{U}) = \mathbb{U}$. If one conjunct of a conjunction is undetermined and the other is true, the conjunction is also undetermined—after all, depending on how the future contingent conjunct turns out, the conjunction might turn out to be true, and it might turn out to be false. So

$$\widetilde{\land}(\mathbb{T}, \mathbb{U}) = \widetilde{\land}(\mathbb{U}, \mathbb{T}) = \mathbb{U}.$$

If the other conjunct is false, however, it cannot turn out true, so

$$\widetilde{\land}(\mathbb{F}, \mathbb{U}) = \widetilde{\land}(\mathbb{F}, \mathbb{U}) = \mathbb{F}.$$

The other values (if the arguments are settled truth values, $\mathbb{T}$ or $\mathbb{F}$, are like in classical logic.

For the conditional, the situation is a little trickier. Suppose $q$ is a future contingent statement. If $p$ is false, then $p \to q$ will be true, regardless of how $q$ turns out, so we should set $\widetilde{\to}(\mathbb{F}, \mathbb{U}) = \mathbb{T}$. And if $p$ is true, then $q \to p$ will be true, regardless of what $q$ turns out to be, so $\widetilde{\to}(\mathbb{U}, \mathbb{T}) = \mathbb{T}$. If $p$ is true,

[1] Łukasiewicz here uses "possible" in a way that is uncommon today, namely to mean possible but not necessary.

then $p \to q$ might turn out to be true or false, so $\widetilde{\to}(\mathbb{T}, \mathbb{U}) = \mathbb{U}$. Similarly, if $p$ is false, then $q \to p$ might turn out to be true or false, so $\widetilde{\to}(\mathbb{U}, \mathbb{F}) = \mathbb{U}$. This leaves the case where $p$ and $q$ are both future contingents. On the basis of the motivation, we should really assign $\mathbb{U}$ in this case. However, this would make $\varphi \to \varphi$ *not* a tautology. Łukasiewicz had not trouble giving up $\varphi \lor \neg\varphi$ and $\neg(\varphi \land \neg\varphi)$, but balked at giving up $\varphi \to \varphi$. So he stipulated $\widetilde{\to}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

**Definition thr.1.** Three-valued Łukasiewicz logic is defined using the matrix: mvl:thr:luk: def:lukasiewicz

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\to$.

2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.

3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.

4. Truth functions are given by the following tables:

| $\widetilde{\neg}$ | |
|---|---|
| $\mathbb{T}$ | $\mathbb{F}$ |
| $\mathbb{U}$ | $\mathbb{U}$ |
| $\mathbb{F}$ | $\mathbb{T}$ |

| $\widetilde{\land}_{\mathbf{Ł}_3}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |
|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |
| $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{F}$ |

| $\widetilde{\lor}_{\mathbf{Ł}_3}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |
|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{U}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |

| $\widetilde{\to}_{\mathbf{Ł}_3}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |
|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ |
| $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{U}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |

As can easily be seen, any formula $\varphi$ containing only $\neg$, $\land$, and $\lor$ will take the truth value $\mathbb{U}$ if all its propositional variables are assigned $\mathbb{U}$. So for instance, the classical tautologies $p \lor \neg p$ and $\neg(p \land \neg p)$ are not tautologies in $\mathbf{Ł}_3$, since $\overline{\mathfrak{v}}(\varphi) = \mathbb{U}$ whenever $\mathfrak{v}(p) = \mathbb{U}$.

On valuations where $\mathfrak{v}(p) = \mathbb{T}$ or $\mathbb{F}$, $\overline{\mathfrak{v}}(\varphi)$ will coincide with its classical truth value.

**Proposition thr.2.** *If $\mathfrak{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$ for all $p$ in $\varphi$, then $\overline{\mathfrak{v}}_{\mathbf{Ł}_3}(\varphi) = \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi)$.*

**Problem thr.1.** Suppose we define $\overline{\mathfrak{v}}(\varphi \leftrightarrow \psi) = \overline{\mathfrak{v}}((\varphi \to \psi) \land (\psi \to \varphi))$ in $\mathbf{Ł}_3$. What truth table would $\leftrightarrow$ have?

Many classical tautologies *are* also tautologies in $\mathbf{Ł}_3$, e.g, $\neg p \to (p \to q)$. Just like in classical logic, we can use truth tables to verify this:

| $p$ | $q$ | $\neg$ | $p$ | $\rightarrow$ | $(p$ | $\rightarrow$ | $q)$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{U}$ |
| $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{F}$ |
| $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ |
| $\mathbb{U}$ | $\mathbb{F}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{U}$ | $\mathbb{U}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{F}$ | $\mathbb{U}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{U}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ | $\mathbb{T}$ | $\mathbb{F}$ |

**Problem thr.2.** Show that the following are tautologies in **Ł**$_3$:

1. $p \rightarrow (q \rightarrow p)$
2. $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
3. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

(In (2) and (3), take $\varphi \leftrightarrow \psi$ as an abbreviation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, or refer to your solution to Problem thr.1.)

**Problem thr.3.** Show that the following classical tautologies are not tautologies in **Ł**$_3$:

1. $(\neg p \wedge p) \rightarrow q)$
2. $((p \rightarrow q) \rightarrow p) \rightarrow p$
3. $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$

One might therefore perhaps think that although not all classical tautologies are tautologies in **Ł**$_3$, they should at least take either the value $\mathbb{T}$ or the value $\mathbb{U}$ on every valuation. This is not the case. A counterexample is given by

$$\neg(p \rightarrow \neg p) \vee \neg(\neg p \rightarrow p)$$

which is $\mathbb{F}$ if $p$ is $\mathbb{U}$.

**Problem thr.4.** Which of the following relations hold in Łukasiewicz logic? Give a truth table for each.

1. $p, p \rightarrow q \vDash q$
2. $\neg\neg p \vDash p$
3. $p \wedge q \vDash p$
4. $p \vDash p \wedge p$
5. $p \vDash p \vee q$

Łukasiewicz hoped to build a logic of possibility on the basis of his three-valued system, by introducing a one-place connective $\Diamond\varphi$ (for "$\varphi$ is possible") and a corresponding $\Box\varphi$ (for "$\varphi$ is necessary"):

| $\widetilde{\Diamond}$ | | | $\widetilde{\Box}$ | |
|---|---|---|---|---|
| $\mathbb{T}$ | $\mathbb{T}$ | | $\mathbb{T}$ | $\mathbb{T}$ |
| $\mathbb{U}$ | $\mathbb{T}$ | | $\mathbb{U}$ | $\mathbb{F}$ |
| $\mathbb{F}$ | $\mathbb{F}$ | | $\mathbb{F}$ | $\mathbb{F}$ |

In other words, $p$ is possible iff it is not already settled as false; and $p$ is necessary iff it is already settled as true.

**Problem thr.5.** Show that $\Box p \leftrightarrow \neg\Diamond\neg p$ and $\Diamond p \leftrightarrow \neg\Box\neg p$ are tautologies in $\mathbf{Ł}_3$, extended with the truth tables for $\Box$ and $\Diamond$.

However, the shortcomings of this proposed modal logic soon became evident: However things turn out, $p \wedge \neg p$ can never turn out to be true. So even if it is not now settled (and therefore undetermined), it should count as impossible, i.e., $\neg\Diamond(p \wedge \neg p)$ should be a tautology. However, if $\mathfrak{v}(p) = \mathbb{U}$, then $\overline{\mathfrak{v}}(\neg\Diamond(p \wedge \neg p)) = \mathbb{U}$. Although Łukasiewicz was correct that two truth values will not be enough to accommodate modal distinctions such as possiblity and necessity, introducing a third truth value is also not enough.

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## Bibliography