

thr.1 Kleene logics

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sec

Stephen Kleene introduced two three-valued logics motivated by a logic in which truth values are thought of the outcomes of computational procedures: a procedure may yield \mathbb{T} or \mathbb{F} , but it may also fail to terminate. In that case the corresponding truth value is undefined, represented by the truth value \mathbb{U} .

To compute the negation of a proposition φ , you would first compute the value of φ , and then return the opposite of the result. If the computation of φ does not terminate, then the entire procedure does not either: so the negation of \mathbb{U} is \mathbb{U} .

To compute a conjunction $\varphi \wedge \psi$, there are two options: one can first compute φ , then ψ , and then the result would be \mathbb{T} if the outcome of both is \mathbb{T} , and \mathbb{F} otherwise. If either computation fails to halt, the entire procedure does as well. So in this case, the if one conjunct is undefined, the conjunction is as well. The same goes for disjunction.

However, if we can evaluate φ and ψ in parallel, we can do better. Then, if one of the two procedures halts and returns \mathbb{F} , we can stop, as the answer must be false. So in that case a conjunction with one false conjunct is false, even if the other conjunct is undefined. Similarly, when computing a disjunction in parallel, we can stop once the procedure for one of the two disjuncts has returned true: then the disjunction must be true. So in this case we can know what the outcome of a compound claim is, even if one of the components is undefined. On this interpretation, we might read \mathbb{U} as “unknown” rather than “undefined.”

The two interpretations give rise to Kleene’s strong and weak logic. The conditional is defined as equivalent to $\neg\varphi \vee \psi$.

Definition thr.1. *Strong Kleene logic \mathbf{Ks}* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

\approx	\mathbb{F}	\approx	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	$\tilde{\wedge}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}

$\tilde{\vee}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{Ks}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}

Definition thr.2. *Weak Kleene logic \mathbf{Kw}* is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. Truth functions are given by the following tables:

\approx		$\tilde{\wedge}_{\mathbf{Kw}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{U}	\mathbb{F}

$\tilde{\vee}_{\mathbf{Kw}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{Kw}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{U}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{T}

Proposition thr.3. \mathbf{Ks} and \mathbf{Kw} have no tautologies.

Proof. If $\mathbf{v}(p) = \mathbb{U}$ for all propositional variables p , then any formula φ will have truth value $\bar{\mathbf{v}}(\varphi) = \mathbb{U}$, since

$$\approx(\mathbb{U}) = \tilde{\vee}(\mathbb{U}, \mathbb{U}) = \tilde{\wedge}(\mathbb{U}, \mathbb{U}) = \tilde{\rightarrow}(\mathbb{U}, \mathbb{U}) = \mathbb{U}$$

in both logics. As $\mathbb{U} \notin V^+$ for either \mathbf{Ks} or \mathbf{Kw} , on this valuation, φ will not be designated. \square

Although both weak and strong Kleene logic have no tautologies, they have non-trivial consequence relations.

Problem thr.1. Which of the following relations hold in (a) strong and (b) weak Kleene logic? Give a truth table for each.

1. $p, p \rightarrow q \vDash q$
2. $p \vee q, \neg p \vDash q$
3. $p \wedge q \vDash p$
4. $p \vDash p \wedge p$
5. $p \vDash p \vee q$

Dmitry Bochvar interpreted \mathbb{U} as “meaningless” and attempted to use it to solve paradoxes such as the Liar paradox by stipulating that paradoxical sentences take the value \mathbb{U} . He introduced a logic which is essentially weak Kleene logic extended by additional connectives, two of which are “external negation” and the “is undefined” operator:

\sim	
T	F
U	T
F	T

$\tilde{+}$	
T	F
U	T
F	F

Problem thr.2. Can you define \sim in Bochvar's logic in terms of \neg and $+$, i.e., find a formula with only the propositional variable p and not involving \sim which always takes the same truth value as $\sim p$? Give a truth table to show you're right.

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Bibliography