thr.1 Gödel logics

mvl:thr:god:sec Kurt Gödel introduced a sequence of n-valued logics that each contain all formulas valid in intuitionistic logic, and are contained in classical logic. Here is the first interesting one:

mvl:thr:god:defn:goedel **Definition thr.1.** 3-*valued Gödel logic* $\mathbf{G}$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\wedge$, $\vee$, $\to$.

2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.

3. $\mathbb{T}$ is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.

4. For $\bot$, we have $\widetilde{\bot} = \mathbb{F}$. Truth functions for the remaining connectives are given by the following tables:

$\widetilde{\neg}_{\mathbf{G}}$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$

$\widetilde{\wedge}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{U}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$\widetilde{\vee}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{U}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$

$\widetilde{\to}_{\mathbf{G}}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{U}$	$\mathbb{F}$
$\mathbb{U}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$

You'll notice that the truth tables for $\wedge$ and $\vee$ are the same as in Łukasiewicz and strong Kleene logic, but the truth tables for $\neg$ and $\to$ differ for each. In Gödel logic, $\widetilde{\neg}(\mathbb{U}) = \mathbb{F}$. In contrast to Łukasiewicz logic and Kleene logic, $\widetilde{\to}(\mathbb{U}, \mathbb{F}) = \mathbb{F}$; in contrast to Kleene logic (but as in Łukasiewicz logic), $\widetilde{\to}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

As the connection to intuitionistic logic alluded to above suggests, $\mathbf{G}_3$ is close to intuitionistic logic. All intuitionistic truths are tautologies in $\mathbf{G}_3$, and many classical tautologies that are not valid intuitionistically also fail to be tautologies in $\mathbf{G}_3$. For instance, the following are not tautologies:

$$\begin{array}{ll} p \vee \neg p & (p \to q) \to (\neg p \vee q) \\ \neg\neg p \to p & \neg(p \wedge q) \to (\neg p \vee \neg q) \\ & ((p \to q) \to p) \to p \end{array}$$

However, not every tautology of $\mathbf{G}_3$ is also intuitionistically valid, e.g., $(p \to q) \vee (q \to p)$.

Problem thr.1. Give a truth table to show that $(p \to q) \vee (q \to p)$ is a tautology of $\mathbf{G}_3$.

Problem thr.2. Give truth tables that show that the following are not tautologies of $\mathbf{G}_3$

$$(p \to q) \to (\neg p \lor q)$$
$$\neg(p \land q) \to (\neg p \lor \neg q)$$
$$((p \to q) \to p) \to p$$

Problem thr.3. Which of the following relations hold in Gödel logic? Give a truth table for each.

1. $p, p \to q \vDash q$
2. $p \lor q, \neg p \vDash q$
3. $p \land q \vDash p$
4. $p \vDash p \land p$
5. $p \vDash p \lor q$

Photo Credits

Bibliography