Kurt Gödel introduced a sequence of $n$-valued logics that each contain all formulas valid in intuitionistic logic, and are contained in classical logic. Here is the first interesting one:

**Definition thr.1.** 3-valued Gödel logic $G$ is defined using the matrix:

1. The standard propositional language $L_0$ with $\bot$, $\neg$, $\land$, $\lor$, $\rightarrow$.
2. The set of truth values $V = \{T, U, F\}$.
3. $T$ is the only designated value, i.e., $V^+ = \{T\}$.
4. For $\bot$, we have $\bot = F$. Truth functions for the remaining connectives are given by the following tables:

<table>
<thead>
<tr>
<th>$\neg_G$</th>
<th>$\land_G$</th>
<th>$\lor_G$</th>
<th>$\rightarrow_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

You’ll notice that the truth tables for $\land$ and $\lor$ are the same as in Łukasiewicz and strong Kleene logic, but the truth tables for $\neg$ and $\rightarrow$ differ for each. In Gödel logic, $\neg(\bot) = F$. In contrast to Łukasiewicz logic and Kleene logic, $\neg(\bot, U) = F$; in contrast to Kleene logic (but as in Łukasiewicz logic), $\neg(\bot, \bot) = T$.

As the connection to intuitionistic logic alluded to above suggests, $G_3$ is close to intuitionistic logic. All intuitionistic truths are tautologies in $G_3$, and many classical tautologies that are not valid intuitionistically also fail to be tautologies in $G_3$. For instance, the following are not tautologies:

$p \lor \neg p$ \hspace{1cm} $(p \rightarrow q) \rightarrow (\neg p \lor q)$

$\neg\neg p \rightarrow p$ \hspace{1cm} $\neg(\neg p \land \neg q) \rightarrow (p \lor q)$

$(p \rightarrow q) \rightarrow p$ \hspace{1cm} $\neg(\neg p \lor q) \rightarrow (p \land \neg q)$

However, not every tautology of $G_3$ is also intuitionistically valid, e.g., $\neg\neg p \lor \neg p$ or $(p \rightarrow q) \lor (q \rightarrow p)$. 
Problem thr.1. Give truth tables to show that the following are tautologies of $G_3$:

\[ \neg
\neg p \lor \neg p \]
\[ (p \rightarrow q) \lor (q \rightarrow p) \]
\[ \neg(p \land q) \rightarrow (\neg p \lor \neg q) \]
\[ (p \rightarrow q) \lor (q \rightarrow r) \lor (r \rightarrow s) \]

Problem thr.2. Give truth tables that show that the following are not tautologies of $G_3$:

\[ (p \rightarrow q) \rightarrow (\neg p \lor q) \]
\[ \neg(\neg p \land \neg q) \rightarrow (p \lor q) \]
\[ ((p \rightarrow q) \rightarrow p) \rightarrow p \]
\[ \neg(p \rightarrow q) \rightarrow (p \land \neg q) \]

Problem thr.3. Which of the following relations hold in Gödel logic? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $p \lor q, \neg p \models q$
3. $p \land q \models p$
4. $p \models p \land p$
5. $p \models p \lor q$

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Bibliography