

thr.1 Gödel logics

mvl:thr:god:sec Kurt Gödel introduced a sequence of n -valued logics that each contain all formulas valid in intuitionistic logic, and are contained in classical logic. Here is the first interesting one:

mvl:thr:god:defn:goedel **Definition thr.1.** *3-valued Gödel logic* \mathbf{G} is defined using the matrix:

1. The standard propositional language \mathcal{L}_0 with $\perp, \neg, \wedge, \vee, \rightarrow$.
2. The set of truth values $V = \{\mathbb{T}, \mathbb{U}, \mathbb{F}\}$.
3. \mathbb{T} is the only designated value, i.e., $V^+ = \{\mathbb{T}\}$.
4. For \perp , we have $\tilde{\perp} = \mathbb{F}$. Truth functions for the remaining connectives are given by the following tables:

$\tilde{\neg}_{\mathbf{G}}$		$\tilde{\wedge}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{F}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{F}	\mathbb{F}	\mathbb{F}	\mathbb{F}

$\tilde{\vee}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}	$\tilde{\rightarrow}_{\mathbf{G}}$	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{T}	\mathbb{U}	\mathbb{F}
\mathbb{U}	\mathbb{T}	\mathbb{U}	\mathbb{U}	\mathbb{U}	\mathbb{T}	\mathbb{T}	\mathbb{F}
\mathbb{F}	\mathbb{T}	\mathbb{U}	\mathbb{F}	\mathbb{F}	\mathbb{T}	\mathbb{T}	\mathbb{T}

You'll notice that the truth tables for \wedge and \vee are the same as in Łukasiewicz and strong Kleene logic, but the truth tables for \neg and \rightarrow differ for each. In Gödel logic, $\tilde{\neg}(\mathbb{U}) = \mathbb{F}$. In contrast to Łukasiewicz logic and Kleene logic, $\tilde{\rightarrow}(\mathbb{U}, \mathbb{F}) = \mathbb{F}$; in contrast to Kleene logic (but as in Łukasiewicz logic), $\tilde{\rightarrow}(\mathbb{U}, \mathbb{U}) = \mathbb{T}$.

As the connection to intuitionistic logic alluded to above suggests, \mathbf{G}_3 is close to intuitionistic logic. All intuitionistic truths are tautologies in \mathbf{G}_3 , and many classical tautologies that are not valid intuitionistically also fail to be tautologies in \mathbf{G}_3 . For instance, the following are not tautologies:

$$\begin{array}{ll}
 p \vee \neg p & (p \rightarrow q) \rightarrow (\neg p \vee q) \\
 \neg \neg p \rightarrow p & \neg(\neg p \wedge \neg q) \rightarrow (p \vee q) \\
 ((p \rightarrow q) \rightarrow p) \rightarrow p & \neg(p \rightarrow q) \rightarrow (p \wedge \neg q)
 \end{array}$$

However, not every tautology of \mathbf{G}_3 is also intuitionistically valid, e.g., $\neg \neg p \vee \neg p$ or $(p \rightarrow q) \vee (q \rightarrow p)$.

Problem thr.1. Give truth tables to show that the following are tautologies of \mathbf{G}_3 :

$$\begin{aligned} & \neg\neg p \vee \neg p \\ & (p \rightarrow q) \vee (q \rightarrow p) \\ & \neg(p \wedge q) \rightarrow (\neg p \vee \neg q) \\ & (p \rightarrow q) \vee (q \rightarrow r) \vee (r \rightarrow s) \end{aligned}$$

Problem thr.2. Give truth tables that show that the following are not tautologies of \mathbf{G}_3

$$\begin{aligned} & (p \rightarrow q) \rightarrow (\neg p \vee q) \\ & \neg(\neg p \wedge \neg q) \rightarrow (p \vee q) \\ & ((p \rightarrow q) \rightarrow p) \rightarrow p \\ & \neg(p \rightarrow q) \rightarrow (p \wedge \neg q) \end{aligned}$$

Problem thr.3. Which of the following relations hold in Gödel logic? Give a truth table for each.

1. $p, p \rightarrow q \models q$
2. $p \vee q, \neg p \models q$
3. $p \wedge q \models p$
4. $p \models p \wedge p$
5. $p \models p \vee q$

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Bibliography