

## syn.1 Many-valued logics as sublogics of $\mathbf{C}$

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sec

The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in  $\{\mathbb{T}, \mathbb{F}\}$  agrees with the classical truth functions. Specifically, in these logics, if  $x \in \{\mathbb{T}, \mathbb{F}\}$ , then  $\tilde{\neg}_{\mathbf{L}}(x) = \tilde{\neg}_{\mathbf{C}}(x)$ , and for  $\star$  any one of  $\wedge, \vee, \rightarrow$ , if  $x, y \in \{\mathbb{T}, \mathbb{F}\}$ , then  $\tilde{\star}_{\mathbf{L}}(x, y) = \tilde{\star}_{\mathbf{C}}(x, y)$ . In other words, the truth functions for  $\neg, \wedge, \vee, \rightarrow$  restricted to  $\{\mathbb{T}, \mathbb{F}\}$  are exactly the classical truth functions.

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prop:mul-cl

**Proposition syn.1.** *Suppose that a many-valued logic  $\mathbf{L}$  contains the connectives  $\neg, \wedge, \vee, \rightarrow$  in its language,  $\mathbb{T}, \mathbb{F} \in V$ , and its truth functions satisfy:*

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1.  $\tilde{\neg}_{\mathbf{L}}(x) = \tilde{\neg}_{\mathbf{C}}(x)$  if  $x = \mathbb{T}$  or  $x = \mathbb{F}$ ;

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2.  $\tilde{\wedge}_{\mathbf{L}}(x, y) = \tilde{\wedge}_{\mathbf{C}}(x, y)$ ,

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prop:lor

3.  $\tilde{\vee}_{\mathbf{L}}(x, y) = \tilde{\vee}_{\mathbf{C}}(x, y)$ ,

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prop:lif

4.  $\tilde{\rightarrow}_{\mathbf{L}}(x, y) = \tilde{\rightarrow}_{\mathbf{C}}(x, y)$ , if  $x, y \in \{\mathbb{T}, \mathbb{F}\}$ .

Then, for any valuation  $\mathbf{v}$  into  $V$  such that  $\mathbf{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$ ,  $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$ .

*Proof.* By induction on  $\varphi$ .

1. If  $\varphi \equiv p$  is atomic, we have  $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbf{v}(p) = \bar{\mathbf{v}}_{\mathbf{C}}(\varphi)$ .

2. If  $\varphi \equiv \neg B$ , we have

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi)) && \text{by ??} \\ &= \tilde{\neg}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by inductive hypothesis} \\ &= \tilde{\neg}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi)) && \text{by assumption (1),} \\ & && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi) \in \{\mathbb{T}, \mathbb{F}\}, \\ &= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by ??}. \end{aligned}$$

3. If  $\varphi \equiv (\psi \wedge \chi)$ , we have

$$\begin{aligned} \bar{\mathbf{v}}_{\mathbf{L}}(\varphi) &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{L}}(\psi), \bar{\mathbf{v}}_{\mathbf{L}}(\chi)) && \text{by ??} \\ &= \tilde{\wedge}_{\mathbf{L}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by inductive hypothesis} \\ &= \tilde{\wedge}_{\mathbf{C}}(\bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi)) && \text{by assumption (2),} \\ & && \text{since } \bar{\mathbf{v}}_{\mathbf{C}}(\psi), \bar{\mathbf{v}}_{\mathbf{C}}(\chi) \in \{\mathbb{T}, \mathbb{F}\}, \\ &= \bar{\mathbf{v}}_{\mathbf{C}}(\varphi) && \text{by ??}. \end{aligned}$$

The cases where  $\varphi \equiv (\psi \vee \chi)$  and  $\varphi \equiv (\psi \rightarrow \chi)$  are similar.  $\square$

**Corollary syn.2.** *If a many-valued logic satisfies the conditions of [Proposition syn.1](#),  $\mathbb{T} \in V^+$  and  $\mathbb{F} \notin V^+$ , then  $\models_{\mathbf{L}} \subseteq \models_{\mathbf{C}}$ , i.e., if  $\Gamma \models_{\mathbf{L}} \psi$  then  $\Gamma \models_{\mathbf{C}} \psi$ . In particular, every tautology of  $\mathbf{L}$  is also a classical tautology.*

*Proof.* We prove the contrapositive. Suppose  $\Gamma \not\models_{\mathbf{C}} \psi$ . Then there is some valuation  $\mathbf{v}: \text{At}_0 \rightarrow \{\mathbb{T}, \mathbb{F}\}$  such that  $\bar{\mathbf{v}}_{\mathbf{C}}(\varphi) = \mathbb{T}$  for all  $\varphi \in \Gamma$  and  $\bar{\mathbf{v}}_{\mathbf{C}}(\psi) = \mathbb{F}$ . Since  $\mathbb{T}, \mathbb{F} \in V$ , the valuation  $\mathbf{v}$  is also a valuation for  $\mathbf{L}$ . By Proposition syn.1,  $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) = \mathbb{T}$  for all  $\varphi \in \Gamma$  and  $\bar{\mathbf{v}}_{\mathbf{L}}(\psi) = \mathbb{F}$ . Since  $\mathbb{T} \in V^+$  and  $\mathbb{F} \notin V^+$  that means  $\mathbf{v} \models_{\mathbf{L}} \Gamma$  and  $\mathbf{v} \not\models_{\mathbf{L}} \psi$ , i.e.,  $\Gamma \not\models_{\mathbf{L}} \psi$ .  $\square$

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## Bibliography