syn.1 Many-valued logics as sublogics of C

mvl:syn:sub:sec The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in $\{\mathbb{T}, \mathbb{F}\}$ agrees with the classical truth functions. Specifically, in these logics, if $x \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$, and for $\star$ any one of $\land$, $\lor$, $\to$, if $x, y \in \{\mathbb{T}, \mathbb{F}\}$, then $\widetilde{\star}_{\mathbf{L}}(x, y) = \widetilde{\star}_{\mathbf{C}}(x, y)$. In other words, the truth functions for $\neg$, $\land$, $\lor$, $\to$ restricted to $\{\mathbb{T}, \mathbb{F}\}$ are exactly the classical truth functions.

mvl:syn:sub:prop:mvl-cl **Proposition syn.1.** *Suppose that a many-valued logic* $\mathbf{L}$ *contains the connectives* $\neg$, $\land$, $\lor$, $\to$ *in its language,* $\mathbb{T}, \mathbb{F} \in V$*, and its truth functions satisfy:*

mvl:syn:sub:prop:not 1. $\widetilde{\neg}_{\mathbf{L}}(x) = \widetilde{\neg}_{\mathbf{C}}(x)$ *if* $x = \mathbb{T}$ *or* $x = \mathbb{F}$*;*

mvl:syn:sub:prop:land 2. $\widetilde{\land}_{\mathbf{L}}(x, y) = \widetilde{\land}_{\mathbf{C}}(x, y)$,

mvl:syn:sub:prop:lor 3. $\widetilde{\lor}_{\mathbf{L}}(x, y) = \widetilde{\lor}_{\mathbf{C}}(x, y)$,

mvl:syn:sub:prop:lif 4. $\widetilde{\to}_{\mathbf{L}}(x, y) = \widetilde{\to}_{\mathbf{C}}(x, y)$, *if* $x, y \in \{\mathbb{T}, \mathbb{F}\}$.

Then, for any valuation $\mathfrak{v}$ *into* V *such that* $\mathfrak{v}(p) \in \{\mathbb{T}, \mathbb{F}\}$, $\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) = \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi)$.

Proof. By induction on φ.

1. If $\varphi \equiv p$ is atomic, we have $\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) = \mathfrak{v}(p) = \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi)$.

2. If $\varphi \equiv \neg B$, we have

$$\begin{aligned}
\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) &= \widetilde{\neg}_{\mathbf{L}}(\overline{\mathfrak{v}}_{\mathbf{L}}(\psi)) && \text{by ??} \\
&= \widetilde{\neg}_{\mathbf{L}}(\overline{\mathfrak{v}}_{\mathbf{C}}(\psi)) && \text{by inductive hypothesis} \\
&= \widetilde{\neg}_{\mathbf{C}}(\overline{\mathfrak{v}}_{\mathbf{C}}(\psi)) && \text{by assumption (1),} \\
& && \text{since } \overline{\mathfrak{v}}_{\mathbf{C}}(\psi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi) && \text{by ??.}
\end{aligned}$$

3. If $\varphi \equiv (\psi \land \chi)$, we have

$$\begin{aligned}
\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) &= \widetilde{\land}_{\mathbf{L}}(\overline{\mathfrak{v}}_{\mathbf{L}}(\psi), \overline{\mathfrak{v}}_{\mathbf{L}}(\chi)) && \text{by ??} \\
&= \widetilde{\land}_{\mathbf{L}}(\overline{\mathfrak{v}}_{\mathbf{C}}(\psi), \overline{\mathfrak{v}}_{\mathbf{C}}(\chi)) && \text{by inductive hypothesis} \\
&= \widetilde{\land}_{\mathbf{C}}(\overline{\mathfrak{v}}_{\mathbf{C}}(\psi), \overline{\mathfrak{v}}_{\mathbf{C}}(\chi)) && \text{by assumption (2),} \\
& && \text{since } \overline{\mathfrak{v}}_{\mathbf{C}}(\psi), \overline{\mathfrak{v}}_{\mathbf{C}}(\chi) \in \{\mathbb{T}, \mathbb{F}\}, \\
&= \overline{\mathfrak{v}}_{\mathbf{C}}(\varphi) && \text{by ??.}
\end{aligned}$$

The cases where $\varphi \equiv (\psi \lor \chi)$ and $\varphi \equiv (\psi \to \chi)$ are similar. □

Corollary syn.2. *If a many-valued logic satisfies the conditions of Proposition syn.1,* $\mathbb{T} \in V^+$ *and* $\mathbb{F} \notin V^+$*, then* $\vDash_{\mathbf{L}} \subseteq \vDash_{\mathbf{C}}$*, i.e., if* $\Gamma \vDash_{\mathbf{L}} \psi$ *then* $\Gamma \vDash_{\mathbf{C}} \psi$*. In particular, every tautology of* $\mathbf{L}$ *is also a classical tautology.*

Proof. We prove the contrapositive. Suppose $\Gamma \nvDash_{\mathbf{C}} \psi$. Then there is some valuation $\mathfrak{v}\colon \mathrm{At}_0 \to \{\mathbb{T}, \mathbb{F}\}$ such that $\overline{\mathfrak{v}}_{\mathbf{C}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\overline{\mathfrak{v}}_{\mathbf{C}}(\psi) = \mathbb{F}$. Since $\mathbb{T}, \mathbb{F} \in V$, the valuation $\mathfrak{v}$ is also a valuation for $\mathbf{L}$. By Proposition syn.1, $\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) = \mathbb{T}$ for all $\varphi \in \Gamma$ and $\overline{\mathfrak{v}}_{\mathbf{L}}(\psi) = \mathbb{F}$. Since $\mathbb{T} \in V^+$ and $\mathbb{F} \notin V^+$ that means $\mathfrak{v} \vDash_{\mathbf{L}} \Gamma$ and $\mathfrak{v} \nvDash_{\mathbf{L}} \psi$, i.e., $\Gamma \nvDash_{\mathbf{L}} \psi$. $\square$

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Bibliography