The usual many-valued logics are all defined using matrices in which the value of a truth-function for arguments in \( \{T, F\} \) agrees with the classical truth functions. Specifically, in these logics, if \( x \in \{T, F\} \), then \( \neg_L(x) = \neg_C(x) \), and for any one of \( \land, \lor, \to \), if \( x, y \in \{T, F\} \), then \( \land_L(x, y) = \land_C(x, y) \). In other words, the truth functions for \( \neg, \land, \lor, \to \) restricted to \( \{T, F\} \) are exactly the classical truth functions.

**Proposition syn.1.** Suppose that a many-valued logic \( L \) contains the connectives \( \neg, \land, \lor, \to \) in its language, \( T, F \in V \), and its truth functions satisfy:

1. \( \neg_L(x) = \neg_C(x) \) if \( x = T \) or \( x = F \);
2. \( \land_L(x, y) = \land_C(x, y) \),
3. \( \lor_L(x, y) = \lor_C(x, y) \), if \( x, y \in \{T, F\} \).

Then, for any valuation \( v \) into \( V \) such that \( v(p) \in \{T, F\} \), \( v_L(\varphi) = v_C(\varphi) \).

**Proof.** By induction on \( \varphi \).

1. If \( \varphi \equiv p \) is atomic, we have \( v_L(\varphi) = v(p) = v_C(\varphi) \).

2. If \( \varphi \equiv \neg B \), we have
   
   \[
   v_L(\varphi) = \neg_L(v_L(\psi)) \quad \text{ by ??} \\
   = \neg_L(v_C(\psi)) \quad \text{ by inductive hypothesis} \\
   = \neg_C(v_C(\psi)) \quad \text{ by assumption (1), since } v_C(\psi) \in \{T, F\}, \\
   = v_C(\varphi) \quad \text{ by ??}.
   \]

3. If \( \varphi \equiv (\psi \land \chi) \), we have
   
   \[
   v_L(\varphi) = \land_L(v_L(\psi), v_L(\chi)) \quad \text{ by ??} \\
   = \land_L(v_C(\psi), v_C(\chi)) \quad \text{ by inductive hypothesis} \\
   = \land_C(v_C(\psi), v_C(\chi)) \quad \text{ by assumption (2), since } v_C(\psi), v_C(\chi) \in \{T, F\}, \\
   = v_C(\varphi) \quad \text{ by ??}.
   \]

The cases where \( \varphi \equiv (\psi \lor \chi) \) and \( \varphi \equiv (\psi \to \chi) \) are similar.

**Corollary syn.2.** If a many-valued logic satisfies the conditions of Proposition syn.1, \( T \in V^+ \) and \( F \notin V^+ \), then \( \models_L \psi \) if and only if \( \models_C \psi \). In particular, every tautology of \( L \) is also a classical tautology.
Proof. We prove the contrapositive. Suppose $\Gamma \not\models_C \psi$. Then there is some valuation $v: \text{At}_0 \to \{T, F\}$ such that $v_C(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_C(\psi) = F$. Since $T, F \in V$, the valuation $v$ is also a valuation for $L$. By Proposition syn.1, $v_L(\varphi) = T$ for all $\varphi \in \Gamma$ and $v_L(\psi) = F$. Since $T \in V^+$ and $F \notin V^+$ that means $v \models_L \Gamma$ and $v \not\models_L \psi$, i.e., $\Gamma \not\models_L \psi$. \hfill \Box

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