

## syn.1 Introduction

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In classical logic, we deal with **formulas** that are built from **propositional variables** using the propositional connectives  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$ , and  $\leftrightarrow$ . When we define a semantics for classical logic, we do so using the two truth values  $\mathbb{T}$  and  $\mathbb{F}$ . We interpret **propositional variables** in a **valuation**  $\mathbf{v}$ , which assigns these truth values  $\mathbb{T}$ ,  $\mathbb{F}$  to the **propositional variables**. Any **valuation** then determines a truth value  $\bar{\mathbf{v}}(\varphi)$  for any **formula**  $\varphi$ , and **A formula** is satisfied in a **valuation**  $\mathbf{v}$ ,  $\mathbf{v} \models \varphi$ , iff  $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$ .

Many-valued logics are generalizations of classical two-valued logic by allowing more truth values than just  $\mathbb{T}$  and  $\mathbb{F}$ . So in many-valued logic, a **valuation**  $\mathbf{v}$  is a function assigning to every **propositional variable**  $p$  one of a range of possible truth values. We'll generally call the set of allowed truth values  $V$ . Classical logic is a many-valued logic where  $V = \{\mathbb{T}, \mathbb{F}\}$ , and the truth value  $\bar{\mathbf{v}}(\varphi)$  is computed using the familiar characteristic truth tables for the connectives.

Once we add additional truth values, we have more than one natural option for how to compute  $\bar{\mathbf{v}}(\varphi)$  for the connectives we read as “and,” “or,” “not,” and “if—then.” So a many-valued logic is determined not just by the set of truth values, but also by the *truth functions* we decide to use for each connective. Once these are selected for a many-valued logic  $\mathbf{L}$ , however, the truth value  $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi)$  is uniquely determined by the valuation, just like in classical logic. Many-valued logics, like classical logic, are *truth functional*.

With this semantic building blocks in hand, we can go on to define the analogs of the semantic concepts of tautology, entailment, and satisfiability. In classical logic, a **formula** is a tautology if its truth value  $\bar{\mathbf{v}}(\varphi) = \mathbb{T}$  for any  $\mathbf{v}$ . In many-valued logic, we have to generalize this a bit as well. First of all, there is no requirement that the set of truth values  $V$  contains  $\mathbb{T}$ . For instance, some many-valued logics use numbers, such as all rational numbers between 0 and 1 as their set of truth values. In such a case, 1 usually plays the role of  $\mathbb{T}$ . In other logics, not just one but several truth values do. So, we require that every many-valued logic have a set  $V^+$  of *designated values*. We can then say that a **formula** is satisfied in a **valuation**  $\mathbf{v}$ ,  $\mathbf{v} \models_{\mathbf{L}} \varphi$ , iff  $\bar{\mathbf{v}}_{\mathbf{L}}(\varphi) \in V^+$ . A **formula**  $\varphi$  is a tautology of the logic,  $\models_{\mathbf{L}} \varphi$ , iff  $\bar{\mathbf{v}}(\varphi) \in V^+$  for any  $\mathbf{v}$ . And, finally, we say that  $\varphi$  is entailed by a set of **formulas**,  $\Gamma \models_{\mathbf{L}} \varphi$ , if every **valuation** that satisfies all the **formulas** in  $\Gamma$  also satisfies  $\varphi$ .

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## Bibliography