syn.1 Introduction

mvl:syn:int:sec In classical logic, we deal with formulas that are built from propositional variables using the propositional connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. When we define a semantics for classical logic, we do so using the two truth values $\mathbb{T}$ and $\mathbb{F}$. We interpret propositional variables in a valuation $\mathfrak{v}$, which assigns these truth values $\mathbb{T}$, $\mathbb{F}$ to the propositional variables. Any valuation then determines a truth value $\overline{\mathfrak{v}}(\varphi)$ for any formula φ, and A formula is satisfied in a valuation $\mathfrak{v}$, $\mathfrak{v} \vDash \varphi$, iff $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$.

Many-valued logics are generalizations of classical two-valued logic by allowing more truth values than just $\mathbb{T}$ and $\mathbb{F}$. So in many-valued logic, a valuation $\mathfrak{v}$ is a function assigning to every propositional variable p one of a range of possible truth values. We'll generally call the set of allowed truth values V. Classical logic is a many-valued logic where $V = \{\mathbb{T}, \mathbb{F}\}$, and the truth value $\overline{\mathfrak{v}}(\varphi)$ is computed using the familiar characteristic truth tables for the connectives.

Once we add additional truth values, we have more than one natural option for how to compute $\overline{\mathfrak{v}}(\varphi)$ for the connectives we read as "and," "or," "not," and "if—then." So a many-valued logic is determined not just by the set of truth values, but also by the *truth functions* we decide to use for each connective. Once these are selected for a many-valued logic $\mathbf{L}$, however, the truth value $\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi)$ is uniquely determined by the valuation, just like in classical logic. Many-valued logics, like classical logic, are *truth functional.*

With this semantic building blocks in hand, we can go on to define the analogs of the semantic concepts of tautology, entailment, and satisfiability. In classical logic, a formula is a tautology if its truth value $\overline{\mathfrak{v}}(\varphi) = \mathbb{T}$ for any $\mathfrak{v}$. In many-valued logic, we have to generalize this a bit as well. First of all, there is no requirement that the set of truth values V contains $\mathbb{T}$. For instance, some many-valued logics use numbers, such as all rational numbers between 0 and 1 as their set of truth values. In such a case, 1 usually plays the rule of $\mathbb{T}$. In other logics, not just one but several truth values do. So, we require that every many-valued logic have a set V^+ of *designated values.* We can then say that a formula is satisfied in a valuation $\mathfrak{v}$, $\mathfrak{v} \vDash_{\mathbf{L}} \varphi$, iff $\overline{\mathfrak{v}}_{\mathbf{L}}(\varphi) \in V^+$. A formula φ is a tautology of the logic, $\vDash_{\mathbf{L}} \varphi$, iff $\overline{\mathfrak{v}}(\varphi) \in V^+$ for any $\mathfrak{v}$. And, finally, we say that φ is entailed by a set of formulas, $\Gamma \vDash_{\mathbf{L}} \varphi$, if every valuation that satisfies all the formulas in Γ also satisfies φ.

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Bibliography