Chapter udf

Infinite-valued Logics

inf.1 Introduction

The number of truth values of a matrix need not be finite. An obvious choice for a set of infinitely many truth values is the set of rational numbers between 0 and 1, \( V_\infty = [0,1] \cap \mathbb{Q} \), i.e.,

\[
V_\infty = \{ \frac{n}{m} : n, m \in \mathbb{N} \text{ and } n \leq m \}.
\]

When considering this infinite truth value set, it is often useful to also consider the subsets

\[
V_m = \{ \frac{n}{m-1} : n \in \mathbb{N} \text{ and } n \leq m \}
\]

For instance, \( V_5 \) is the set with 5 evenly spaced truth values,

\[
V_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}.
\]

In logics based on these truth value sets, usually only 1 is designated, i.e., \( V^+ = \{1\} \). In other words, we let 1 play the role of (absolute) truth, 0 as absolute falsity, but formulas may take any intermediate value in \( V \).

One can also consider the set \( V_{[0,1]} = [0,1] \) of all real numbers between 0 and 1, or other infinite subsets of \([0,1]\), however. Logics with this truth value set are often called fuzzy.

inf.2 Lukasiewicz logic

This is a short “stub” of a section on infinite-valued Lukasiewicz logic.
**Definition inf.1.** Infinite-valued Lukasiewicz logic $\mathbb{L}_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\neg$, $\land$, $\lor$, $\rightarrow$.
2. The set of truth values $V_\infty$.
3. $1$ is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:
   \[
   \begin{align*}
   \neg_L(x) &= 1 - x \\
   \land_L(x, y) &= \min(x, y) \\
   \lor_L(x, y) &= \max(x, y) \\
   \rightarrow_L(x, y) &= \min(1, 1 - (x - y)) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (x - y) & \text{otherwise.} \end{cases}
   \end{align*}
   \]

$m$-valued Lukasiewicz logic is defined the same, except $V = V_m$.

**Proposition inf.2.** The logic $\mathbb{L}_3$ defined by ?? is the same as $\mathbb{L}_3$ defined by **Definition inf.1.**

**Proof.** This can be seen by comparing the truth tables for the connectives given in ?? with the truth tables determined by the equations in **Definition inf.1.**

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>$\land_{L_3}$</th>
<th>$\lor_{L_3}$</th>
<th>$\rightarrow_{L_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Proposition inf.3.** If $\Gamma \vdash_{\mathbb{L}_\infty} \psi$ then $\Gamma \vdash_{\mathbb{L}_m} \psi$ for all $m \geq 2$.

**Proof.** Exercise.

**Problem inf.1.** Prove **Proposition inf.3.**

In fact, the converse holds as well.

Infinite-valued Lukasiewicz logic is the most popular fuzzy logic. In the fuzzy logic literature, the conditional is often defined as $\neg \varphi \lor \psi$. The result would be an infinite-valued strong Kleene logic.

**Problem inf.2.** Show that $(p \rightarrow q) \lor (q \rightarrow p)$ is a tautology of $\mathbb{L}_\infty$. 
Definition inf.4. Infinite-valued Gödel logic $G_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot, \neg, \wedge, \vee, \to$.
2. The set of truth values $V_\infty$.
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:
   
   $\overline{\bot} = 0$
   
   $\overline{\neg}_G(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$
   
   $\overline{\wedge}_G(x, y) = \min(x, y)$
   
   $\overline{\vee}_G(x, y) = \max(x, y)$
   
   $\overline{\rightarrow}_G(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$

$m$-valued Gödel logic is defined the same, except $V = V_m$.

Proposition inf.5. The logic $G_3$ defined by ?? is the same as $G_3$ defined by Definition inf.4.

Proof. This can be seen by comparing the truth tables for the connectives given in ?? with the truth tables determined by the equations in Definition inf.4:

<table>
<thead>
<tr>
<th>$\overline{\neg}_G$</th>
<th>$\overline{\wedge}_G$</th>
<th>$\overline{\vee}_G$</th>
<th>$\overline{\rightarrow}_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bot$</td>
<td>$\top$</td>
<td>$\bot$</td>
<td>$\top$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Proposition inf.6. If $\Gamma \models_{G_\infty} \psi$ then $\Gamma \models_{G_m} \psi$ for all $m \geq 2$.

Proof. Exercise.
Problem inf.3. Prove Proposition inf.6.

In fact, the converse holds as well.

Like $G_3$, $G_\infty$ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of $G_\infty$:

\[
\begin{align*}
    p \lor \neg p & \quad (p \rightarrow q) \rightarrow (\neg p \lor q) \\
    \neg \neg p \rightarrow p & \quad \neg (\neg p \land \neg q) \rightarrow (p \lor q) \\
    ((p \rightarrow q) \rightarrow p) \rightarrow p & \quad \neg (p \rightarrow q) \rightarrow (p \land \neg q)
\end{align*}
\]

The example of an intuitionistically invalid formula that is nevertheless a tautology of $G_3$, $(p \rightarrow q) \lor (q \rightarrow p)$, is also a tautology in $G_\infty$. In fact, $G_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ is added. This was shown by Michael Dummett, and so $G_\infty$ is often referred to as Gödel–Dummett logic LC.

Problem inf.4. Show that $(p \rightarrow q) \lor (q \rightarrow p)$ is a tautology of $G_\infty$.

Problem inf.5. Show that $(p \rightarrow q) \lor (q \rightarrow r) \lor (r \rightarrow s)$, which is a tautology of $G_3$, is not a tautology of $G_\infty$.

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Bibliography