Chapter udf

Infinite-valued Logics

inf.1 Introduction

The number of truth values of a matrix need not be finite. An obvious choice for a set of infinitely many truth values is the set of rational numbers between 0 and 1, $V_\infty = [0, 1] \cap \mathbb{Q}$, i.e.,

$$V_\infty = \{ \frac{n}{m} : n, m \in \mathbb{N} \text{ and } n \leq m \}.$$

When considering this infinite truth value set, it is often useful to also consider the subsets

$$V_m = \{ \frac{n}{m-1} : n \in \mathbb{N} \text{ and } n \leq m \}.$$

For instance, $V_5$ is the set with 5 evenly spaced truth values,

$$V_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}.$$

In logics based on these truth value sets, usually only 1 is designated, i.e., $V^+ = \{1\}$. In other words, we let 1 play the role of (absolute) truth, 0 as absolute falsity, but formulas may take any intermediate value in $V$.

One can also consider the set $V_{[0,1]} = [0, 1]$ of all real numbers between 0 and 1, or other infinite subsets of $[0, 1]$, however. Logics with this truth value set are often called fuzzy.

inf.2 Lukasiewicz logic

This is a short “stub” of a section on infinite-valued Lukasiewicz logic.
Definition inf.1. Infinite-valued Lukasiewicz logic $L_\infty$ is defined using the matrix:

1. The standard propositional language $L_0$ with $\neg$, $\land$, $\lor$, $\rightarrow$.
2. The set of truth values $V_\infty$.
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:
   
   $\neg_L(x) = 1 - x$
   $\land_L(x, y) = \min(x, y)$
   $\lor_L(x, y) = \max(x, y)$
   $\rightarrow_L(x, y) = \min(1, 1 - (x - y)) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - (x - y) & \text{otherwise.} \end{cases}$

$m$-valued Lukasiewicz logic is defined the same, except $V = V_m$.

Proposition inf.2. The logic $L_3$ defined by ? is the same as $L_3$ defined by Definition inf.1.

Proof. This can be seen by comparing the truth tables for the connectives given in ? with the truth tables determined by the equations in Definition inf.1:

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<tr>
<th>$\neg$</th>
<th>$\land_{L_3}$</th>
<th>$\lor_{L_3}$</th>
<th>$\rightarrow_{L_3}$</th>
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</thead>
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Proposition inf.3. If $\Gamma \models_{L_\infty} \psi$ then $\Gamma \models_{L_m} \psi$ for all $m \geq 2$.

Proof. Exercise.

Problem inf.1. Prove Proposition inf.3.

In fact, the converse holds as well.

Infinite-valued Lukasiewicz logic is the most popular fuzzy logic. In the fuzzy logic literature, the conditional is often defined as $\neg \psi \lor \psi$. The result would be an infinite-valued strong Kleene logic.

Problem inf.2. Show that $(p \rightarrow q) \lor (q \rightarrow p)$ is a tautology of $L_\infty$. 
This is a short “stub” of a section on infinite-valued Gödel logic.

**Definition inf.4.** Infinite-valued Gödel logic $\mathbf{G}_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot, \neg, \land, \lor, \rightarrow$.
2. The set of truth values $V_\infty$.
3. $1$ is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:
   
   $\neg_{\mathbf{G}}(x) = \begin{cases} 
   1 & \text{if } x = 0 \\
   0 & \text{otherwise}
   \end{cases}$

   $\land_{\mathbf{G}}(x, y) = \min(x, y)$

   $\lor_{\mathbf{G}}(x, y) = \max(x, y)$

   $\rightarrow_{\mathbf{G}}(x, y) = \begin{cases} 
   1 & \text{if } x \leq y \\
   y & \text{otherwise}.
   \end{cases}$

$m$-valued Gödel logic is defined the same, except $V = V_m$.

**Proposition inf.5.** The logic $\mathbf{G}_3$ defined by ?? is the same as $\mathbf{G}_3$ defined by Definition inf.4.

**Proof.** This can be seen by comparing the truth tables for the connectives given in ?? with the truth tables determined by the equations in Definition inf.4:

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<tr>
<th>$\neg_{\mathbf{G}}$</th>
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<th>$\land_{\mathbf{G}}$</th>
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**Proposition inf.6.** If $\Gamma \vdash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vdash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.

**Proof.** Exercise.
Problem inf.3. Prove Proposition inf.6.

In fact, the converse holds as well.

Like $G_3$, $G_\infty$ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of $G_\infty$:

\[
\begin{align*}
p \lor \neg p & \quad (p \rightarrow q) \rightarrow (\neg p \lor q) \\
\neg \neg p \rightarrow p & \quad \neg (\neg p \land \neg q) \rightarrow (p \lor q) \\
((p \rightarrow q) \rightarrow p) \rightarrow p & \quad \neg (p \rightarrow q) \rightarrow (p \land \neg q)
\end{align*}
\]

The example of an intuitionistically invalid formula that is nevertheless a tautology of $G_3$, $(p \rightarrow q) \lor (q \rightarrow p)$, is also a tautology in $G_\infty$. In fact, $G_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$ is added. This was shown by Michael Dummett, and so $G_\infty$ is often referred to as Gödel–Dummett logic LC.

Problem inf.4. Show that $(p \rightarrow q) \lor (q \rightarrow p)$ is a tautology of $G_\infty$.

Problem inf.5. Show that $(p \rightarrow q) \lor (q \rightarrow r) \lor (r \rightarrow s)$, which is a tautology of $G_3$, is not a tautology of $G_\infty$.

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