inf.1 Gödel logics

mvl:inf:god:sec

This is a short "stub" of a section on infinite-valued Gödel logic.

mvl:inf:god:def:goedel **Definition inf.1.** Infinite-valued Gödel logic $\mathbf{G}_\infty$ is defined using the matrix:

1. The standard propositional language $\mathcal{L}_0$ with $\bot$, $\neg$, $\wedge$, $\vee$, $\to$.
2. The set of truth values V_∞.
3. 1 is the only designated value, i.e., $V^+ = \{1\}$.
4. Truth functions are given by the following functions:

$$\begin{aligned}
\widetilde{\bot} &= 0 \\
\widetilde{\neg}_{\mathbf{G}}(x) &= \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \\
\widetilde{\wedge}_{\mathbf{G}}(x, y) &= \min(x, y) \\
\widetilde{\vee}_{\mathbf{G}}(x, y) &= \max(x, y) \\
\widetilde{\to}_{\mathbf{G}}(x, y) &= \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}
\end{aligned}$$

m-valued Gödel logic is defined the same, except $V = V_m$.

Proposition inf.2. *The logic $\mathbf{G}_3$ defined by **??** is the same as $\mathbf{G}_3$ defined by Definition inf.1.*

Proof. This can be seen by comparing the truth tables for the connectives given in **??** with the truth tables determined by the equations in Definition inf.1:

$\widetilde{\neg}_{\mathbf{G}_3}$	
1	0
1/2	0
0	1

$\widetilde{\wedge}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\widetilde{\vee}_{\mathbf{G}}$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\widetilde{\to}_{\mathbf{G}}$	1	1/2	0
1	1	1/2	0
1/2	1	1	0
0	1	1	1

□

mvl:inf:god:prop:god-infty-m **Proposition inf.3.** *If $\Gamma \vDash_{\mathbf{G}_\infty} \psi$ then $\Gamma \vDash_{\mathbf{G}_m} \psi$ for all $m \geq 2$.*

Proof. Exercise. □

Problem inf.1. Prove Proposition inf.3.

In fact, the converse holds as well.

Like $\mathbf{G}_3$, $\mathbf{G}_\infty$ has all intuitionistically valid formulas as tautologies, and the same examples of non-tautologies are non-tautologies of $\mathbf{G}_\infty$:

$$
\begin{array}{ll}
p \lor \neg p & (p \to q) \to (\neg p \lor q) \\
\neg\neg p \to p & \neg(\neg p \land \neg q) \to (p \lor q) \\
((p \to q) \to p) \to p & \neg(p \to q) \to (p \land \neg q)
\end{array}
$$

The example of an intuitionistically invalid formula that is nevertheless a tautology of $\mathbf{G}_3$, $(p \to q) \lor (q \to p)$, is also a tautology in $\mathbf{G}_\infty$. In fact, $\mathbf{G}_\infty$ can be characterized as intuitionistic logic to which the schema $(\varphi \to \psi) \lor (\psi \to \varphi)$ is added. This was shown by Michael Dummett, and so $\mathbf{G}_\infty$ is often referred to as Gödel–Dummett logic **LC**.

Problem inf.2. Show that $(p \to q) \lor (q \to p)$ is a tautology of $\mathbf{G}_\infty$.

Problem inf.3. Show that $(p \to q) \lor (q \to r) \lor (r \to s)$, which is a tautology of $\mathbf{G}_3$, is not a tautology of $\mathbf{G}_\infty$.

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Bibliography