syn.1 Terms as α-Equivalence Classes

From now on, we will consider terms up to α-equivalence. That means when we write a term, we mean its α-equivalance class it is in. For example, we write $\lambda a.\, \lambda b.\, ac$ for the set of all terms α-equivalent to it, such as $\lambda a.\, \lambda b.\, ac$, $\lambda b.\, \lambda a.\, bc$, etc.

Also, while in previous sections letters such as N, Q are used to denote a term, from now on we use them to denote a class, and it is these classes instead of terms that will be our subjects of study in what follows. Letters such as x, y continues to denote a variable.

We also adopt the notation $\underline{M}$ to denote an arbitrary element of the class M, and $\underline{M}_0, \underline{M}_1, etc.$ if we need more than one.

We reuse the notations from terms to simplify our wording. We have following definition on classes:

Definition syn.1.   1. $\lambda x.\, N$ is defined as the class containing $\lambda x.\, \underline{N}$.

    2. PQ is defined to be the class containing $\underline{P}\,\underline{Q}$.

It is not hard to see that they are well defined, because α-conversion is compatible.

Definition syn.2. The *free variables* of an α-equivalence class M, or $FV(M)$, is defined to be $FV(\underline{M})$.

This is well defined since $FV(\underline{M}_0) = FV(\underline{M}_1)$, as shown in **??**.
We also reuse the notation for substitution into classes:

Definition syn.3. The *substitution* of R for y in M, or $M[R/y]$, is defined to be $\underline{M}_{R/y}$, for any $\underline{M}$ and $\underline{R}$ making the substition defined.

This is also well defined as shown in **??**.
Note how this definition significantly simplifies our reasoning. For example:

$$\lambda x.\, x[y/x] = \tag{1}$$
$$= \lambda z.\, z[y/x] \tag{2}$$
$$= \lambda z.\, z[y/x] \tag{3}$$
$$= \lambda z.\, z \tag{4}$$

eq. (1) is undefined if we still regard it as substitution on terms; but as mentioned earlier, we now consider it a substitution on classes, which is why eq. (2) can happen: we can replace $\lambda x.\, x$ with $\lambda z.\, z$ because they belong to the same class.

For the same reason, from now on we will assume that the representatives we choose always satisfy the conditions needed for substitution. For example, when we see $\lambda x.\, N[R/y]$, we will assume the representative $\lambda x.\, N$ is chosen so that $x \neq y$ and $x \notin FV(R)$.

Since it is a bit strange to call $\lambda x.\, x$ a "class", let's call them Λ-terms (or simply "terms" in the rest of the part) from now on, to distinguish them from λ-terms that we are familiar with.

> We cannot say goodbye to terms yet: the whole definition of Λ-terms is based on λ-terms, and we haven't provided a method to define functions on Λ-terms, which means all such functions have to be first defined on λ-terms, and then "projected" to Λ-terms, as we did for substitutions. However we assume the reader can intuitively understand how we can define functions on Λ-terms.

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Bibliography