

syn.1 Substitution

Free variables are references to environment variables, thus it makes sense to actually use a specific value in the place of a free variable. For example, we may want to replace f in $\lambda x. fx$ with a specific term, like the identity function $\lambda y. y$. This results in $\lambda x. (\lambda y. y)x$. The process of replacing free variables with lambda terms is called substitution.

Definition syn.1 (Substitution). The *substitution* of a term N for a variable x in a term M , $M[N/x]$, is defined inductively by:

- $x[N/x] = N$.
- $y[N/x] = y$ if $x \neq y$.
- $PQ[N/x] = (P[N/x])(Q[N/x])$.
- $(\lambda y. P)[N/x] = \lambda y. P[N/x]$, if $x \neq y$ and $y \notin \text{FV}(N)$, otherwise undefined.

In **Definition syn.1(4)**, we require $x \neq y$ because we don't want to replace *bound* occurrences of the variable x in M by N . For example, if we compute the substitution $\lambda x. x[y/x]$, the result should not be $\lambda x. y$ but simply $\lambda x. x$.

When substituting N for x in $\lambda y. P$, we also require that $y \notin \text{FV}(N)$. For example, we cannot substitute y for x in $\lambda y. x$, i.e., $\lambda y. x[y/x]$, because it would result in $\lambda y. y$, a term that stands for the function that accepts an argument and returns it directly. But the term $\lambda y. x$ stands for a function that always returns the term x (or whatever x refers to). So the result we actually want is a function that accepts an argument, drop it, and returns the environment variable y . To do this properly, we would first have to “rename” the bound variable y .

Problem syn.1. What is the result of the following substitutions?

- $\lambda y. x(\lambda w. vwx)[(uv)/x]$
- $\lambda y. x(\lambda x. x)[(\lambda y. xy)/x]$
- $y(\lambda v. xv)[(\lambda y. vy)/x]$

Theorem syn.2. If $x \notin \text{FV}(M)$, then $\text{FV}(M[N/x]) = \text{FV}(M)$, if the left-hand side is defined.

Proof. By induction on the formation of M .

- M is a variable: exercise.
- M is of the form (PQ) : exercise.

3. M is of the form $\lambda y. P$, and since $\lambda y. P[N/x]$ is defined, it has to be $\lambda y. P[N/x]$. Then $P[N/x]$ has to be defined; also, $x \neq y$ and $x \notin \text{FV}(Q)$. Then:

$$\begin{aligned}
 \text{FV}(\lambda y. P[N/x]) &= \\
 &= \text{FV}(\lambda y. P[N/x]) && \text{by (4)} \\
 &= \text{FV}(P[N/x]) \setminus \{y\} && \text{by ?????} \\
 &= \text{FV}(P) \setminus \{y\} && \text{by inductive hypothesis} \quad \square \\
 &= \text{FV}(\lambda y. P) && \text{by ?????}
 \end{aligned}$$

Problem syn.2. Complete the proof of **Theorem syn.2**.

Theorem syn.3. *If $x \in \text{FV}(M)$, then $\text{FV}(M[N/x]) = (\text{FV}(M) \setminus \{x\}) \cup \text{FV}(N)$, provided the left hand is defined.* lam:syn:sub:
thm:info

Proof. By induction on the formation of M .

1. M is a variable: exercise.
2. M is of the form PQ : Since $(PQ)[N/y]$ is defined, it has to be $(P[N/x])(Q[N/x])$ with both substitution defined. Also, since $x \in \text{FV}(PQ)$, either $x \in \text{FV}(P)$ or $x \in \text{FV}(Q)$ or both. The rest is left as an exercise.
3. M is of the form $\lambda y. P$. Since $\lambda y. P[N/x]$ is defined, it has to be $\lambda y. P[N/x]$, with $P[N/x]$ defined, $x \neq y$ and $y \notin \text{FV}(N)$; also, since $y \in \text{FV}(\lambda x. P)$, we have $y \in \text{FV}(P)$ too. Now:

$$\begin{aligned}
 \text{FV}((\lambda y. P)[N/x]) &= \\
 &= \text{FV}(\lambda y. P[N/x]) \\
 &= \text{FV}(P[N/x]) \setminus \{y\} \\
 &= ((\text{FV}(P) \setminus \{y\}) \cup (\text{FV}(N) \setminus \{x\})) && \text{by inductive hypothesis} \quad \square \\
 &= (\text{FV}(P) \setminus \{x, y\}) \cup \text{FV}(N) && x \notin \text{FV}(N) \\
 &= (\text{FV}(\lambda y. P) \setminus \{x\}) \cup \text{FV}(N)
 \end{aligned}$$

Problem syn.3. Complete the proof of **Theorem syn.3**.

Theorem syn.4. *$x \notin \text{FV}(M[N/x])$, if the right-hand side is defined and $x \notin \text{FV}(N)$.* lam:syn:sub:
thm:clr

Proof. Exercise. □

Problem syn.4. Prove **Theorem syn.4**.

lam:syn:sub: **Theorem syn.5.** *thm:inv* If $M[y/x]$ is defined and $y \notin \text{FV}(M)$, then $M[y/x][x/y] = M$.

Proof. By induction on the formation of M .

1. M is a variable z : Exercise.
2. M is of the form (PQ) . Then:

$$\begin{aligned}(PQ)[y/x][x/y] &= ((P[y/x])(Q[y/x]))[x/y] \\ &= (P[y/x][x/y])(Q[y/x][x/y]) \\ &= (PQ) \text{ by inductive hypothesis}\end{aligned}$$

3. M is of the form $\lambda z. N$. Because $\lambda z. N[y/x]$ is defined, we know that $z \neq y$. So:

$$\begin{aligned}(\lambda z. N)[y/x][x/y] &= (\lambda z. N[y/x])[x/y] \\ &= \lambda z. N[y/x][x/y] \\ &= \lambda z. N \text{ by inductive hypothesis} \quad \square\end{aligned}$$

Problem syn.5. Complete the proof of **Theorem syn.5**.

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Bibliography