syn.1 Substitution

lam:syn:sub:sec

Free variables are references to environment variables, thus it makes sense to actually use a specific value in the place of a free variable. For example, we may want to replace f in $\lambda x. f x$ with a specific term, like the identity function $\lambda y. y$. This results in $\lambda x. (\lambda y. y)x$. The process of replacing free variables with lambda terms is called substitution. explanation

lam:syn:sub:defn:substitution

Definition syn.1 (Substitution). The *substitution* of a term N for a variable x in a term M, $M[N/x]$, is defined inductively by:

1. $x[N/x] = N$. (lam:syn:sub:defn:substitution-1)
2. $y[N/x] = y$ if $x \neq y$. (lam:syn:sub:defn:substitution-2)
3. $PQ[N/x] = (P[N/x])(Q[N/x])$. (lam:syn:sub:def:substitution-3)
4. $(\lambda y. P)[N/x] = \lambda y. P[N/x]$, if $x \neq y$ and $y \notin \mathrm{FV}(N)$, otherwise undefined. (lam:syn:sub:defn:substitution-4)

In Definition syn.1(4), we require $x \neq y$ because we don't want to replace *bound* occurrences of the variable x in M by N. For example, if we compute the substitution $\lambda x. x[y/x]$, the result should not be $\lambda x. y$ but simply $\lambda x. x$. explanation

When substituting N for x in $\lambda y. P$, we also require that $y \notin \mathrm{FV}(N)$. For example, we cannot substitute y for x in $\lambda y. x$, i.e., $\lambda y. x[y/x]$, because it would result in $\lambda y. y$, a term that stands for the function that accepts an argument and returns it directly. But the term $\lambda y. x$ stands for a function that always returns the term x (or whatever x refers to). So the result we actually want is a function that accepts an argument, drop it, and returns the environment variable y. To do this properly, we would first have to "rename" the bound variable y.

Problem syn.1. What is the result of the following substitutions?

1. $\lambda y. x(\lambda w. vwx)[(uv)/x]$
2. $\lambda y. x(\lambda x. x)[(\lambda y. xy)/x]$
3. $y(\lambda v. xv)[(\lambda y. vy)/x]$

lam:syn:sub:thm:notinfv

Theorem syn.2. *If $x \notin \mathrm{FV}(M)$, then $\mathrm{FV}(M[N/x]) = \mathrm{FV}(M)$, if the left-hand side is defined.*

Proof. By induction on the formation of M.

1. M is a variable: exercise.
2. M is of the form (PQ): exercise.

3. M is of the form $\lambda y.\, P$, and since $\lambda y.\, P[N/x]$ is defined, it has to be $\lambda y.\, P[N/x]$. Then $P[N/x]$ has to be defined; also, $x \neq y$ and $x \notin \mathrm{FV}(Q)$. Then:

$$\begin{aligned}
&\mathrm{FV}(\lambda y.\, P[N/x]) = \\
&\qquad\qquad\qquad\qquad = \mathrm{FV}(\lambda y.\, P[N/x]) && \text{by (4)} \\
&\qquad\qquad\qquad\qquad = \mathrm{FV}(P[N/x]) \setminus \{y\} && \text{by } \textbf{????} \\
&\qquad\qquad\qquad\qquad = \mathrm{FV}(P) \setminus \{y\} && \text{by inductive hypothesis} \\
&\qquad\qquad\qquad\qquad = \mathrm{FV}(\lambda y.\, P) && \text{by } \textbf{????}
\end{aligned}$$

□

Problem syn.2. Complete the proof of Theorem syn.2.

Theorem syn.3. *If $x \in \mathrm{FV}(M))$, then $\mathrm{FV}(M[N/x]) = (\mathrm{FV}(M) \setminus \{x\}) \cup \mathrm{FV}(N)$, provided the left hand is defined.* *lam:syn:sub:thm:infv*

Proof. By induction on the formation of M.

1. M is a variable: exercise.

2. M is of the form PQ: Since $(PQ)[N/y]$ is defined, it has to be $(P[N/x])(Q[N/x])$ with both substitution defined. Also, since $x \in \mathrm{FV}(PQ)$, either $x \in \mathrm{FV}(P)$ or $x \in \mathrm{FV}(Q)$ or both. The rest is left as an exercise.

3. M is of the form $\lambda y.\, P$. Since $\lambda y.\, P[N/x]$ is defined, it has to be $\lambda y.\, P[N/x]$, with $P[N/x]$ defined, $x \neq y$ and $y \notin \mathrm{FV}(N)$; also, since $y \in \mathrm{FV}(\lambda x.\, P)$, we have $y \in \mathrm{FV}(P)$ too. Now:

$$\begin{aligned}
&\mathrm{FV}((\lambda y.\, P)[N/x]) = \\
&\quad = \mathrm{FV}(\lambda y.\, P[N/x]) \\
&\quad = \mathrm{FV}(P[N/x]) \setminus \{y\} \\
&\quad = ((\mathrm{FV}(P) \setminus \{y\}) \cup (\mathrm{FV}(N) \setminus \{x\}) && \text{by inductive hypothesis} \\
&\quad = (\mathrm{FV}(P) \setminus \{x, y\}) \cup \mathrm{FV}(N) && x \notin \mathrm{FV}(N) \\
&\quad = (\mathrm{FV}(\lambda y.\, P) \setminus \{x\}) \cup \mathrm{FV}(N)
\end{aligned}$$

□

Problem syn.3. Complete the proof of Theorem syn.3.

Theorem syn.4. *$x \notin \mathrm{FV}(M[N/x])$, if the right-hand side is defined and $x \notin \mathrm{FV}(N)$.* *lam:syn:sub:thm:clr*

Proof. Exercise. □

Problem syn.4. Prove Theorem syn.4.

lam:syn:sub:thm:inv **Theorem syn.5.** *If $M[y/x]$ is defined and $y \notin \mathrm{FV}(M)$, then $M[y/x][x/y] = M$.*

Proof. By induction on the formation of M.

1. M is a variable z: Exercise.
2. M is of the form (PQ). Then:

$$\begin{aligned}(PQ)[y/x][x/y] &= ((P[y/x])(Q[y/x]))[x/y] \\ &= (P[y/x][x/y])(Q[y/x][x/y]) \\ &= (PQ) \text{ by inductive hypothesis}\end{aligned}$$

3. M is of the form $\lambda z.\, N$. Because $\lambda z.\, N[y/x]$ is defined, we know that $z \neq y$. So:

$$\begin{aligned}&(\lambda z.\, N)[y/x][x/y] \\ &\quad = (\lambda z.\, N[y/x])[x/y] \\ &\quad = \lambda z.\, N[y/x][x/y] \\ &\quad = \lambda z.\, N \text{ by inductive hypothesis}\end{aligned}$$

□

Problem syn.5. Complete the proof of Theorem syn.5.

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Bibliography