syn.1 η-conversion

lam:syn:eta: sec

There is another relation on λ terms. In ?? we used the example $\lambda x.\,(fx)$, which accepts an argument and applies f to it. In other words, it is the same function as f: $\lambda x.\,(fx)N$ and fN both reduce to fN. We use η-reduction (and η-extension) to capture this idea.

lam:syn:eta: defn:beredone

Definition syn.1 (η-contraction, $\xrightarrow{\eta}$). *η-contraction* ($\xrightarrow{\eta}$) is the smallest compatible relation on terms satisfying the following condition:

$$\lambda x.\, Mx \xrightarrow{\eta} M \text{ provided } x \notin FV(M)$$

lam:syn:eta: defn:bered

Definition syn.2 ($\beta\eta$-reduction, $\xrightarrow{\beta\eta}\!\!\!\!\rightarrow$). *$\beta\eta$-reduction* ($\xrightarrow{\beta\eta}\!\!\!\!\rightarrow$) is the smallest reflexive, transitive relation on terms containing $\xrightarrow{\beta}$ and $\xrightarrow{\eta}$, i.e., the rules of reflexivity and transitive plus the following two rules:

lam:syn:eta: defn:bered3

1. If $M \xrightarrow{\beta} N$ then $M \xrightarrow{\beta\eta}\!\!\!\!\rightarrow N$.

lam:syn:eta: defn:bered4

2. If $M \xrightarrow{\eta} N$ then $M \xrightarrow{\beta\eta}\!\!\!\!\rightarrow N$.

Definition syn.3. We extend the equivalence relation $=$ with the η-conversion rule:

$$\lambda x.\, fx = f$$

and denote the extended relation as $\overset{\eta}{=}$.

η-equivalence is important because it is related to extensionality of lambda terms:

Definition syn.4 (Extensionality). We extend the equivalence relation $=$ with the (ext) rule:

$$\text{If } Mx = Nx \text{ then } M = N\text{, provided } x \notin FV(MN).$$

and denote the extended relation as $\overset{ext}{=}$.

Roughly speaking, the rule states that two terms, viewed as functions, should be considered equal if they behave the same for the same argument.

We now prove that the η rule provides exactly the extensionality, and nothing else.

Theorem syn.5. *$M \overset{ext}{=} N$ if and only if $M \overset{\eta}{=} N$.*

Proof. First we prove that $\overset{\eta}{=}$ is closed under the extensionality rule. That is, *ext* rule doesn't add anything to $\overset{\eta}{=}$. We then have $\overset{\eta}{=}$ contains $\overset{ext}{=}$, and if $M \overset{ext}{=} N$, then $M \overset{\eta}{=} N$.

To prove $\overset{\eta}{=}$ is closed under *ext*, note that for any $M = N$ derived by the *ext* rule, we have $Mx \overset{\eta}{=} Nx$ as premise. Then we have $\lambda x.\, Mx \overset{\eta}{=} \lambda x.\, Nx$ by a rule of $=$, applying η on both side gives us $M \overset{\eta}{=} N$.

Similarly we prove that the η rule is contained in $\overset{ext}{=}$. For any $\lambda x.\, Mx$ and M with $x \notin FV(M)$, we have that $(\lambda x.\, Mx)x \overset{ext}{=} Mx$, giving us $\lambda x.\, Mx \overset{ext}{=} M$ by the *ext* rule. $\square$

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Bibliography