

syn.1 η -conversion

lam:syn:eta:
sec There is another relation on λ terms. In ?? we used the example $\lambda x.(fx)$, which accepts an argument and applies f to it. In other words, it is the same function as $f: \lambda x.(fx)N$ and fN both reduce to fN . We use η -reduction (and η -extension) to capture this idea.

lam:syn:eta:
defn:beredone **Definition syn.1 (η -contraction, $\xrightarrow{\eta}$).** η -contraction ($\xrightarrow{\eta}$) is the smallest compatible relation on terms satisfying the following condition:

$$\lambda x.Mx \xrightarrow{\eta} M \text{ provided } x \notin FV(M)$$

lam:syn:eta:
defn:bered **Definition syn.2 ($\beta\eta$ -reduction, $\xrightarrow{\beta\eta}$).** $\beta\eta$ -reduction ($\xrightarrow{\beta\eta}$) is the smallest reflexive, transitive relation on terms containing $\xrightarrow{\beta}$ and $\xrightarrow{\eta}$, i.e., the rules of reflexivity and transitive plus the following two rules:

lam:syn:eta:
defn:bered3 1. If $M \xrightarrow{\beta} N$ then $M \xrightarrow{\beta\eta} N$.

lam:syn:eta:
defn:bered4 2. If $M \xrightarrow{\eta} N$ then $M \xrightarrow{\beta\eta} N$.

Definition syn.3. We extend the equivalence relation $=$ with the η -conversion rule:

$$\lambda x.fx = f$$

and denote the extended relation as $\stackrel{\eta}{=}$.

η -equivalence is important because it is related to extensionality of lambda terms:

Definition syn.4 (Extensionality). We extend the equivalence relation $=$ with the (*ext*) rule:

$$\text{If } Mx = Nx \text{ then } M = N, \text{ provided } x \notin FV(MN).$$

and denote the extended relation as $\stackrel{ext}{=}$.

Roughly speaking, the rule states that two terms, viewed as functions, should be considered equal if they behave the same for the same argument.

We now prove that the η rule provides exactly the extensionality, and nothing else.

Theorem syn.5. $M \stackrel{ext}{=} N$ if and only if $M \stackrel{\eta}{=} N$.

Proof. First we prove that $\stackrel{\eta}{=}$ is closed under the extensionality rule. That is, *ext* rule doesn't add anything to $\stackrel{\eta}{=}$. We then have $\stackrel{\eta}{=}$ contains $\stackrel{ext}{=}$, and if $M \stackrel{ext}{=} N$, then $M \stackrel{\eta}{=} N$.

To prove $\stackrel{\eta}{=}$ is closed under *ext*, note that for any $M = N$ derived by the *ext* rule, we have $Mx \stackrel{\eta}{=} Nx$ as premise. Then we have $\lambda x. Mx \stackrel{\eta}{=} \lambda x. Nx$ by a rule of $=$, applying η on both side gives us $M \stackrel{\eta}{=} N$.

Similarly we prove that the η rule is contained in $\stackrel{ext}{=}$. For any $\lambda x. Mx$ and M with $x \notin FV(M)$, we have that $(\lambda x. Mx)x \stackrel{ext}{=} Mx$, giving us $\lambda x. Mx \stackrel{ext}{=} M$ by the *ext* rule. \square

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Bibliography