syn.1 The De Bruijn Index

lam:syn:deb:sec α-Equivalence is very natural, as terms that are α-equivalent "mean the same." In fact, it is possible to give a syntax for lambda terms which does not distinguish terms that can be α-converted to each other. The best known replaces variables by their *De Bruijn index*.

When we write $\lambda x.\, M$, we explicitly state that x is the parameter of the function, so that we can use x in M to refer to this parameter. In the de Bruijn index, however, parameters have no name and reference to them in the function body is denoted by a number denoting the levels of abstraction between them. For example, consider the example of $\lambda x.\, \lambda y.\, yx$: the outer abstraction is on binds the variable x; the inner abstraction binds the variable is y; the sub-term yx lies in the scope of the inner abstraction: there is no abstraction between y and its abstract λy, but one abstract between x and its abstract λx. Thus we write $0\,1$ for yx, and $\lambda.\, \lambda.\, 01$ for the entire term.

Definition syn.1. De Bruijn terms are inductively defines as follows:

1. n, where n is any natural number.
2. PQ, where P and Q are both De Bruijn terms.
3. $\lambda.\, N$, where N is a De Bruijn term.

A formalized translation from ordinary lambda terms to De Bruijn indexed terms is as follows:

Definition syn.2.

$$\begin{aligned} F_\Gamma(x) &= \Gamma(x) \\ F_\Gamma(PQ) &= F_\Gamma(P)F_\Gamma(Q) \\ F_\Gamma(\lambda x.\, N) &= \lambda.\, F_{x,\Gamma}(N) \end{aligned}$$

where Γ is a list of variables indexed from zero, and $\Gamma(x)$ denotes the position of the variable x in Γ. For example, if Γ is x, y, z, then $\Gamma(x)$ is 0 and $\Gamma(z)$ is 2.

x, Γ denotes the list resulted from pushing x to the head of Γ; for instance, continuing the Γ in last example, w, Γ is w, x, y, z.

Recovering a standard lambda term from a de Bruijn term is done as follows:

Definition syn.3.

$$\begin{aligned} G_\Gamma(n) &= \Gamma[n] \\ G_\Gamma(PQ) &= G_\Gamma(P)G_\Gamma(Q) \\ G_\Gamma(\lambda.\, N) &= \lambda x.\, G_{x,\Gamma}(N) \end{aligned}$$

where Γ is again a list of variables indexed from zero, and $\Gamma[n]$ denotes the variable in position n. For example, if Γ is x, y, z, then $\Gamma[1]$ is y.

The variable x in last equation is chosen to be any variable that not in Γ.

Here we give some results without proving them:

Proposition syn.4. *If* $M \xrightarrow{\alpha} M'$, *and* Γ *is any list containing* $\mathrm{FV}(M)$, *then* $F_\Gamma(M) \equiv F_\Gamma(M')$.

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Bibliography