The De Bruijn Index

$\alpha$-Equivalence is very natural, as terms that are $\alpha$-equivalent “mean the same.” In fact, it is possible to give a syntax for lambda terms which does not distinguish terms that can be $\alpha$-converted to each other. The best known replaces variables by their De Bruijn index.

When we write $\lambda x. M$, we explicitly state that $x$ is the parameter of the function, so that we can use $x$ in $M$ to refer to this parameter. In the de Bruijn index, however, parameters have no name and reference to them in the function body is denoted by a number denoting the levels of abstraction between them. For example, consider the example of $\lambda x. \lambda y. yx$: the outer abstraction is on binds the variable $x$; the inner abstraction binds the variable is $y$; the sub-term $yx$ lies in the scope of the inner abstraction: there is no abstraction between $y$ and its abstract $\lambda y$, but one abstract between $x$ and its abstract $\lambda x$. Thus we write $01$ for $yx$, and $\lambda. \lambda. 01$ for the entire term.

**Definition syn.1.** De Bruijn terms are inductively defined as follows:

1. $n$, where $n$ is any natural number.
2. $PQ$, where $P$ and $Q$ are both De Bruijn terms.
3. $\lambda. N$, where $N$ is a De Bruijn term.

A formalized translation from ordinary lambda terms to De Bruijn indexed terms is as follows:

**Definition syn.2.**

$$F_\Gamma(x) = \Gamma(x)$$
$$F_\Gamma(PQ) = F_\Gamma(P)F_\Gamma(Q)$$
$$F_\Gamma(\lambda x. N) = \lambda. F_{x, \Gamma}(N)$$

where $\Gamma$ is a list of variables indexed from zero, and $\Gamma(x)$ denotes the position of the variable $x$ in $\Gamma$. For example, if $\Gamma$ is $x, y, z$, then $\Gamma(x)$ is 0 and $\Gamma(z)$ is 2.

$x, \Gamma$ denotes the list resulted from pushing $x$ to the head of $\Gamma$; for instance, continuing the $\Gamma$ in last example, $w, \Gamma$ is $w, x, y, z$.

Recovering a standard lambda term from a de Bruijn term is done as follows:

**Definition syn.3.**

$$G_\Gamma(n) = \Gamma[n]$$
$$G_\Gamma(PQ) = G_\Gamma(P)G_\Gamma(Q)$$
$$G_\Gamma(\lambda. N) = \lambda x. G_{x, \Gamma}(N)$$

where $\Gamma$ is again a list of variables indexed from zero, and $\Gamma[n]$ denotes the variable in position $n$. For example, if $\Gamma$ is $x, y, z$, then $\Gamma[1]$ is $y$.

The variable $x$ in last equation is chosen to be any variable that not in $\Gamma$.  

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Here we give some results without proving them:

**Proposition syn.4.** If $M \xrightarrow{\alpha} M'$, and $\Gamma$ is any list containing $\text{FV}(M)$, then $F_\Gamma(M) \equiv F_\Gamma(M')$.

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Bibliography