

## int.1 $\beta$ -reduction

lam:int:bet:  
sec When we see  $(\lambda m. (\lambda y. y)m)$ , it is natural to conjecture that it has some connection with  $\lambda m. m$ , namely the second term should be the result of “simplifying” the first. The notion of  $\beta$ -reduction captures this intuition formally.

lam:int:bet:  
defn:betacontr **Definition int.1 ( $\beta$ -contraction,  $\xrightarrow{\beta}$ ).** The  $\beta$ -contraction ( $\xrightarrow{\beta}$ ) is the smallest compatible relation on terms satisfying the following condition:

$$(\lambda x. N)Q \xrightarrow{\beta} N[Q/x]$$

We say  $P$  is  $\beta$ -contracted to  $Q$  if  $P \xrightarrow{\beta} Q$ . A term of the form  $(\lambda x. N)Q$  is called a *redex*.

lam:int:bet:  
prob:def **Problem int.1.** Spell out the equivalent inductive definitions of  $\beta$ -contraction as we did for change of bound variable in ??.

lam:int:bet:  
defn:betared **Definition int.2 ( $\beta$ -reduction,  $\xrightarrow{\beta}$ ).**  $\beta$ -reduction ( $\xrightarrow{\beta}$ ) is the smallest reflexive, transitive relation on terms containing  $\xrightarrow{\beta}$ . We say  $P$  is  $\beta$ -reduced to  $Q$  if  $P \xrightarrow{\beta} Q$ .

We will write  $\rightarrow$  instead of  $\xrightarrow{\beta}$ , and  $\twoheadrightarrow$  instead of  $\xrightarrow{\beta}$  when context is clear.

Informally speaking,  $M \xrightarrow{\beta} N$  if and only if  $M$  can be changed to  $N$  by zero or several steps of  $\beta$ -contraction.

**Definition int.3 ( $\beta$ -normal).** A term that cannot be  $\beta$ -contracted any further is said to be  $\beta$ -normal.

If  $M \xrightarrow{\beta} N$  and  $N$  is  $\beta$ -normal, then we say  $N$  is a *normal form* of  $M$ . One may ask if the normal form of a term is unique, and the answer is yes, as we will see later.

Let us consider some examples.

1. We have

$$\begin{aligned} (\lambda x. xxy)\lambda z. z &\rightarrow (\lambda z. z)(\lambda z. z)y \\ &\rightarrow (\lambda z. z)y \\ &\rightarrow y \end{aligned}$$

2. “Simplifying” a term can actually make it more complex:

$$\begin{aligned} (\lambda x. xxy)(\lambda x. xxy) &\rightarrow (\lambda x. xxy)(\lambda x. xxy)y \\ &\rightarrow (\lambda x. xxy)(\lambda x. xxy)yy \\ &\rightarrow \dots \end{aligned}$$

3. It can also leave a term unchanged:

$$(\lambda x. xx)(\lambda x. xx) \rightarrow (\lambda x. xx)(\lambda x. xx)$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda y. yv)z$$

by contracting the outermost application; and

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda x. zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term,  $zv$ .

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the Church-Rosser property.

digression In general, there is more than one way to  $\beta$ -reduce a term, thus many reduction strategies have been invented, among which the most common is the *natural strategy*. The natural strategy always contracts the *left-most* redex, where the position of a redex is defined as its starting point in the term. The natural strategy has the useful property that a term can be reduced to a normal form by some strategy iff it can be reduced to normal form using the natural strategy. In what follows we will use the natural strategy unless otherwise specified.

**Definition int.4 ( $\beta$ -equivalence, =).**  $\beta$ -Equivalence (=) is the relation inductively defined as follows:

1.  $M = M$ .
2. If  $M = N$ , then  $N = M$ .
3. If  $M = N$ ,  $N = O$ , then  $M = O$ .
4. If  $M = N$ , then  $PM = PN$ .
5. If  $M = N$ , then  $MQ = NQ$ .
6. If  $M = N$ , then  $\lambda x. M = \lambda x. N$ .
7.  $(\lambda x. N)Q = N[Q/x]$ .

The first three rules make the relation an equivalence relation; the next three make it compatible; the last ensures that it contains  $\beta$ -contraction.

Informally speaking, two terms are  $\beta$ -equivalent if and only if one of them can be changed to the other in zero or more steps of  $\beta$ -contraction, or “inverse” of  $\beta$ -contraction. The inverse of  $\beta$ -contraction is defined so that  $M$  inverse- $\beta$ -contracts to  $N$  iff  $N$   $\beta$ -contracts to  $M$ .

Besides the above rules, we will extend the relation with more rules, and denote the extended equivalence relation as  $\stackrel{X}{\equiv}$ , where  $X$  is the extending rule.

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## **Bibliography**