

## ldf.1 Primitive Recursive Functions are $\lambda$ -Definable

lam:ldf:prf:sec Recall that the primitive recursive functions are those that can be defined from the basic functions zero, succ, and  $P_i^n$  by composition and primitive recursion.

lam:ldf:prf:lem:basic **Lemma ldf.1.** *The basic primitive recursive functions zero, succ, and projections  $P_i^n$  are  $\lambda$ -definable.*

*Proof.* They are  $\lambda$ -defined by the following terms:

$$\begin{aligned} \text{Zero} &\equiv \lambda a. \lambda f x. x \\ \text{Succ} &\equiv \lambda a. \lambda f x. f(afx) \\ \text{Proj}_i^n &\equiv \lambda x_0 \dots x_{n-1}. x_i \end{aligned} \quad \square$$

lam:ldf:prf:lem:comp **Lemma ldf.2.** *Suppose the  $k$ -ary function  $f$ , and  $n$ -ary functions  $g_0, \dots, g_{k-1}$ , are  $\lambda$ -definable by terms  $F, G_0, \dots, G_k$ , and  $h$  is defined from them by composition. Then  $H$  is  $\lambda$ -definable*

*Proof.*  $h$  can be  $\lambda$ -defined by the term

$$H \equiv \lambda x_0 \dots x_{n-1}. F(G_0 x_0 \dots x_{n-1}) \dots (G_{k-1} x_0 \dots x_{n-1})$$

We leave verification of this fact as an exercise.  $\square$

**Problem ldf.1.** Complete the proof of **Lemma ldf.2** by showing that  $H\bar{n}_0 \dots \bar{n}_{n-1} \rightarrow \bar{h}(n_0, \dots, n_{n-1})$ .

Note that **Lemma ldf.2** did not require that  $f$  and  $g_0, \dots, g_{k-1}$  are primitive recursive; it is only required that they are total and  $\lambda$ -definable.

lam:ldf:prf:lem:prim **Lemma ldf.3.** *Suppose  $f$  is an  $n$ -ary function and  $g$  is an  $n+2$ -ary function, they are  $\lambda$ -definable by terms  $F$  and  $G$ , and the function  $h$  is defined from  $f$  and  $g$  by primitive recursion. Then  $h$  is also  $\lambda$ -definable.*

*Proof.* Recall that  $h$  is defined by

$$\begin{aligned} h(x_1, \dots, x_n, 0) &= f(x_1, \dots, x_n) \\ h(x_1, \dots, x_n, y + 1) &= h(x_1, \dots, x_n, y, h(x_1, \dots, x_n, y)). \end{aligned}$$

Informally speaking, the primitive recursive definition iterates the application of the function  $h$   $y$  times and applies it to  $f(x_1, \dots, x_n)$ . This is reminiscent of the definition of Church numerals, which is also defined as an iterator.

For simplicity, we give the definition and proof for a single additional argument  $x$ . The function  $h$  is  $\lambda$ -defined by:

$$H \equiv \lambda x. \lambda y. \text{Snd}(yD(\bar{0}, Fx))$$

where

$$D \equiv \lambda p. \langle \text{Succ}(\text{Fst } p), (Gx(\text{Fst } p)(\text{Snd } p)) \rangle$$

The iteration state we maintain is a pair, the first of which is the current  $y$  and the second is the corresponding value of  $h$ . For every step of iteration we create a pair of new values of  $y$  and  $h$ ; after the iteration is done we return the second part of the pair and that's the final  $h$  value. We now prove this is indeed a representation of primitive recursion.

We want to prove that for any  $n$  and  $m$ ,  $H \bar{n} \bar{m} \rightarrow \overline{h(n, m)}$ . To do this we first show that if  $D_n \equiv D[\bar{n}/x]$ , then  $D_n^m \langle \bar{0}, F \bar{n} \rangle \rightarrow \langle \bar{m}, \overline{h(n, m)} \rangle$ . We proceed by induction on  $m$ .

If  $m = 0$ , we want  $D_n^0 \langle \bar{0}, F \bar{n} \rangle \rightarrow \langle \bar{0}, \overline{h(n, 0)} \rangle$ . But  $D_n^0 \langle \bar{0}, F \bar{n} \rangle$  just is  $\langle \bar{0}, F \bar{n} \rangle$ . Since  $F$   $\lambda$ -defines  $f$ , this reduces to  $\langle \bar{0}, f(n) \rangle$ , and since  $f(n) = h(n, 0)$ , this is  $\langle \bar{0}, \overline{h(n, 0)} \rangle$ .

Now suppose that  $D_n^m \langle \bar{0}, F \bar{n} \rangle \rightarrow \langle \bar{m}, \overline{h(n, m)} \rangle$ . We want to show that  $D_n^{m+1} \langle \bar{0}, F \bar{n} \rangle \rightarrow \langle \overline{m+1}, \overline{h(n, m+1)} \rangle$ .

$$\begin{aligned} D_n^{m+1} \langle \bar{0}, F \bar{n} \rangle &\equiv D_n(D_n^m \langle \bar{0}, F \bar{n} \rangle) \\ &\rightarrow D_n \langle \bar{m}, \overline{h(n, m)} \rangle \quad (\text{by IH}) \\ &\equiv (\lambda p. \langle \text{Succ}(\text{Fst } p), (G \bar{n}(\text{Fst } p)(\text{Snd } p)) \rangle) \langle \bar{m}, \overline{h(n, m)} \rangle \\ &\rightarrow \langle \text{Succ}(\text{Fst } \langle \bar{m}, \overline{h(n, m)} \rangle), \\ &\quad (G \bar{n}(\text{Fst } \langle \bar{m}, \overline{h(n, m)} \rangle)(\text{Snd } \langle \bar{m}, \overline{h(n, m)} \rangle)) \rangle \\ &\rightarrow \langle \text{Succ } \bar{m}, (G \bar{n} \bar{m} \overline{h(n, m)}) \rangle \\ &\rightarrow \langle \overline{m+1}, \overline{g(n, m, h(n, m))} \rangle \end{aligned}$$

Since  $g(n, m, h(n, m)) = h(n, m+1)$ , we are done.

Finally, consider

$$\begin{aligned} H \bar{n} \bar{m} &\equiv \lambda x. \lambda y. \text{Snd}(y(\lambda p. \langle \text{Succ}(\text{Fst } p), (Gx(\text{Fst } p)(\text{Snd } p)) \rangle) \langle \bar{0}, Fx \rangle) \\ &\quad \bar{n} \bar{m} \\ &\rightarrow \text{Snd}(\bar{m} \underbrace{(\lambda p. \langle \text{Succ}(\text{Fst } p), (G \bar{n}(\text{Fst } p)(\text{Snd } p)) \rangle)}_{D_n}) \langle \bar{0}, F \bar{n} \rangle \\ &\equiv \text{Snd}(\bar{m} D_n \langle \bar{0}, F \bar{n} \rangle) \\ &\rightarrow \text{Snd}(D_n^m \langle \bar{0}, F \bar{n} \rangle) \\ &\rightarrow \text{Snd} \langle \bar{m}, \overline{h(n, m)} \rangle \\ &\rightarrow \overline{h(n, m)}. \quad \square \end{aligned}$$

**Proposition ldf.4.** *Every primitive recursive function is  $\lambda$ -definable.*

*Proof.* By [Lemma ldf.1](#), all basic functions are  $\lambda$ -definable, and by [Lemma ldf.2](#) and [Lemma ldf.3](#), the  $\lambda$ -definable functions are closed under composition and primitive recursion.  $\square$

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**Bibliography**