

ldf.1 Minimization

lam:ldf:min:
sec

The general recursive functions are those that can be obtained from the basic functions zero, succ, P_i^n by composition, primitive recursion, and regular minimization. To show that all general recursive functions are λ -definable we have to show that any function defined by regular minimization from a λ -definable function is itself λ -definable.

lam:ldf:min:
lem:min

Lemma ldf.1. *If $f(x_1, \dots, x_k, y)$ is regular and λ -definable, then g defined by*

$$g(x_1, \dots, x_k) = \mu y f(x_1, \dots, x_k, y) = 0$$

is also λ -definable.

Proof. Suppose the lambda term F λ -defines the regular function $f(\vec{x}, y)$. To λ -define h we use a search function and a fixpoint combinator:

$$\begin{aligned} \text{Search} &\equiv \lambda g. \lambda f \vec{x} y. \text{IsZero}(f \vec{x} y) y (g \vec{x} (\text{Succ } y)) \\ H &\equiv \lambda \vec{x}. (Y \text{ Search}) F \vec{x} \bar{0}, \end{aligned}$$

where Y is any fixpoint combinator. Informally speaking, Search is a self-referencing function: starting with y , test whether $f \vec{x} y$ is zero: if so, return y , otherwise call itself with Succ y . Thus $(Y \text{ Search}) F \bar{n}_1 \dots \bar{n}_k \bar{0}$ returns the least m for which $f(n_1, \dots, n_k, m) = 0$.

Specifically, observe that

$$(Y \text{ Search}) F \bar{n}_1 \dots \bar{n}_k \bar{m} \rightarrow \bar{m}$$

if $f(n_1, \dots, n_k, m) = 0$, or

$$\rightarrow (Y \text{ Search}) F \bar{n}_1 \dots \bar{n}_k \overline{m + 1}$$

otherwise. Since f is regular, $f(n_1, \dots, n_k, y) = 0$ for some y , and so

$$(Y \text{ Search}) F \bar{n}_1 \dots \bar{n}_k \bar{0} \rightarrow \overline{h(n_1, \dots, n_k)}. \quad \square$$

Proposition ldf.2. *Every general recursive function is λ -definable.*

Proof. By ??, all basic functions are λ -definable, and by ??, ??, and **Lemma ldf.1**, the λ -definable functions are closed under composition, primitive recursion, and regular minimization. \square

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Bibliography