

cr.1 Parallel β -reduction

`lam:cr:pb:` We introduce the notion of *parallel β -reduction*, and prove the it has the
`sec` Church-Rosser property.

`lam:cr:pb:` **Definition cr.1 (parallel β -reduction, $\xRightarrow{\beta}$).** Parallel reduction ($\xRightarrow{\beta}$) of
`defn:bredpar` terms is inductively defined as follows:

- `lam:cr:pb:` 1. $x \xRightarrow{\beta} x$.
`defn:bredpar1`
- `lam:cr:pb:` 2. If $N \xrightarrow{\beta} N'$ then $\lambda x. N \xRightarrow{\beta} \lambda x. N'$.
`defn:bredpar2`
- `lam:cr:pb:` 3. If $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$ then $PQ \xRightarrow{\beta} P'Q'$.
`defn:bredpar3`
- `lam:cr:pb:` 4. If $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$ then $(\lambda x. N)Q \xRightarrow{\beta} N'[Q'/x]$.
`defn:bredpar4`

Parallel β -reduction allows us to reduce any number of redices in a term in one step. It is different from β -reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel β -reduction. For example, the term $(\lambda f. fx)(\lambda y. y)$ can only be parallel β -reduced to itself or to $(\lambda y. y)x$, but not further to x , although it β -reduces to x , because this redex arises only after one step of parallel β -reduction. A second parallel β -reduction step yields x , though.

`lam:cr:pb:` **Theorem cr.2.** $M \xRightarrow{\beta} M$.
`thm:refl`

Proof. Exercise. □

Problem cr.1. Prove **Theorem cr.2**.

`lam:cr:pb:` **Definition cr.3 (β -complete development).** The β -complete development $M^{*\beta}$
`defn:bcd` of M is defined inductively as follows:

$$\text{lam:cr:pb: } x^{*\beta} = x \tag{1}$$

$$\text{defn:bcd1 } (\lambda x. N)^{*\beta} = \lambda x. N^{*\beta} \tag{2}$$

$$\text{lam:cr:pb: } (PQ)^{*\beta} = P^{*\beta}Q^{*\beta} \quad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{3}$$

$$\text{defn:bcd2 } ((\lambda x. N)Q)^{*\beta} = N^{*\beta}[Q^{*\beta}/x] \tag{4}$$

`defn:bcd3`

`defn:bcd4`

The β -complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel β -reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f. fx)(\lambda y. y)$ is $(\lambda y. y)x$, not itself.

This definition has the problem that we haven't introduced how to define functions on (λ -)terms recursively. Will fix in future.

Lemma cr.4. If $M \xRightarrow{\beta} M'$ and $R \xRightarrow{\beta} R'$, then $M[R/y] \xRightarrow{\beta} M'[R'/y]$.

*lam:cr:pb:
lem:comp*

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last step is (1): Exercise.
2. The last step is (2): Then M is $\lambda x. N$ and M' is $\lambda x. N'$, where $N \xRightarrow{\beta} N'$. We want to prove that $(\lambda x. N)[R/y] \xRightarrow{\beta} (\lambda x. N')[R'/y]$, i.e., $\lambda x. N[R/y] \xRightarrow{\beta} \lambda x. N'[R'/y]$. This follows immediately by (2) and the induction hypothesis.
3. The last step is (3): Exercise.
4. The last step is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$. We want to prove that $((\lambda x. N)Q)[R/y] \xRightarrow{\beta} N'[Q'/x][R'/y]$, i.e., $(\lambda x. N[R/y])Q[R/y] \xRightarrow{\beta} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis. \square

Problem cr.2. Complete the proof of Lemma cr.4.

Lemma cr.5. If $M \xRightarrow{\beta} M'$ then $M' \xRightarrow{\beta} M^{*\beta}$.

*lam:cr:pb:
lem:cont*

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last rule is (1): Exercise.
2. The last rule is (2): M is $\lambda x. N$ and M' is $\lambda x. N'$ with $N \xRightarrow{\beta} N'$. We want to show that $\lambda x. N' \xRightarrow{\beta} (\lambda x. N)^{*\beta}$, i.e., $\lambda x. N' \xRightarrow{\beta} \lambda x. N^{*\beta}$ by eq. (2). It follows by (2) and the induction hypothesis.
3. The last rule is (3): M is PQ and M' is $P'Q'$ for some P, Q, P' and Q' , with $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis, we have $P' \xRightarrow{\beta} P^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$.
 - a) If P is $\lambda x. N$ for some x and N , then P' must be $\lambda x. N'$ for some N' with $N \xRightarrow{\beta} N'$. By induction hypothesis we have $N' \xRightarrow{\beta} N^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$. Then $(\lambda x. N')Q' \xRightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$ by (4).
 - b) If P is not a λ -abstract, then $P'Q' \xRightarrow{\beta} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by eq. (3).

4. The last rule is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$ for some x, N, Q, N' , and Q' , with $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we know $N' \xRightarrow{\beta} N^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$. By **Lemma cr.4** we have $N'[Q'/x] \xRightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^{*\beta}$. \square

Problem cr.3. Complete the proof of **Lemma cr.5**.

lam:cr:pb: **Theorem cr.6.** $\xRightarrow{\beta}$ has the Church-Rosser property.
thm:cr

Proof. Immediate from **Lemma cr.5**. \square

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Bibliography