cr.1 Parallel β-reduction

lam:cr:pb:sec We introduce the notion of *parallel β-reduction*, and prove the it has the Church–Rosser property.

lam:cr:pb:defn:bredpar **Definition cr.1 (parallel β-reduction, $\xRightarrow{\beta}$).** Parallel reduction ($\xRightarrow{\beta}$) of terms is inductively defined as follows:

lam:cr:pb:defn:bredpar1
1. $x \xRightarrow{\beta} x$.

lam:cr:pb:defn:bredpar2
2. If $N \xrightarrow{\beta} N'$ then $\lambda x.\, N \xRightarrow{\beta} \lambda x.\, N'$.

lam:cr:pb:defn:bredpar3
3. If $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$ then $PQ \xRightarrow{\beta} P'Q'$.

lam:cr:pb:defn:bredpar4
4. If $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$ then $(\lambda x.\, N)Q \xRightarrow{\beta} N'[Q'/x]$.

Parallel β-reduction allows us to reduce any number of redices in a term in one step. It is different from β-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel β-reduction. For example, the term $(\lambda f.\, fx)(\lambda y.\, y)$ can only be parallel β-reduced to itself or to $(\lambda y.\, y)x$, but not further to x, although it β-reduces to x, because this redex arises only after one step of parallel β-reduction. A second parallel β-reduction step yields x, though.

lam:cr:pb:thm:refl **Theorem cr.2.** $M \xRightarrow{\beta} M$.

Proof. Exercise. ☐

Problem cr.1. Prove Theorem cr.2.

lam:cr:pb:defn:bcd **Definition cr.3 (β-complete development).** The *β-complete development* $M^{*\beta}$ of M is defined inductively as follows:

lam:cr:pb:defn:bcd1
$$x^{*\beta} = x \tag{1}$$
lam:cr:pb:defn:bcd2
$$(\lambda x.\, N)^{*\beta} = \lambda x.\, N^{*\beta} \tag{2}$$
lam:cr:pb:defn:bcd3
$$(PQ)^{*\beta} = P^{*\beta}Q^{*\beta} \qquad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{3}$$
lam:cr:pb:defn:bcd4
$$((\lambda x.\, N)Q)^{*\beta} = N^{*\beta}[Q^{*\beta}/x] \tag{4}$$

The β-complete development of a term, as its name suggests, is a "complete parallel reduction." While for parallel β-reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f.\, fx)(\lambda y.\, y)$ is $(\lambda y.\, y)x$, not itself.

This definition has the problem that we haven't introduced how to define functions on (λ-)terms recursively. Will fix in future.

Lemma cr.4. *If $M \overset{\beta}{\Longrightarrow} M'$ and $R \overset{\beta}{\Longrightarrow} R'$, then $M[R/y] \overset{\beta}{\Longrightarrow} M'[R'/y]$.* *lam:cr:pb:lem:comp*

Proof. By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last step is (1): Exercise.
2. The last step is (2): Then M is $\lambda x.\, N$ and M' is $\lambda x.\, N'$, where $N \overset{\beta}{\Longrightarrow} N'$. We want to prove that $(\lambda x.\, N)[R/y] \overset{\beta}{\Longrightarrow} (\lambda x.\, N')[R'/y]$, i.e., $\lambda x.\, N[R/y] \overset{\beta}{\Longrightarrow} \lambda x.\, N'[R/y]$. This follows immediately by (2) and the induction hypothesis.
3. The last step is (3): Exercise.
4. The last step is (4): M is $(\lambda x.\, N)Q$ and M' is $N'[Q'/x]$. We want to prove that $((\lambda x.\, N)Q)[R/y] \overset{\beta}{\Longrightarrow} N'[Q'/x][R'/y]$, i.e., $(\lambda x.\, N[R/y])Q[R/y] \overset{\beta}{\Longrightarrow} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis. □

Problem cr.2. Complete the proof of Lemma cr.4.

Lemma cr.5. *If $M \overset{\beta}{\Longrightarrow} M'$ then $M' \overset{\beta}{\Longrightarrow} M^{*\beta}$.* *lam:cr:pb:lem:cont*

Proof. By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last rule is (1): Exercise.
2. The last rule is (2): M is $\lambda x.\, N$ and M' is $\lambda x.\, N'$ with $N \overset{\beta}{\Longrightarrow} N'$. We want to show that $\lambda x.\, N' \overset{\beta}{\Longrightarrow} (\lambda x.\, N)^{*\beta}$, i.e., $\lambda x.\, N' \overset{\beta}{\Longrightarrow} \lambda x.\, N^{*\beta}$ by eq. (2). It follows by (2) and the induction hypothesis.
3. The last rule is (3):M is PQ and M' is $P'Q'$ for some P, Q, P' and Q', with $P \overset{\beta}{\Longrightarrow} P'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis, we have $P' \overset{\beta}{\Longrightarrow} P^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$.
   a) If P is $\lambda x.\, N$ for some x and N, then P' must be $\lambda x.\, N'$ for some N' with $N \overset{\beta}{\Longrightarrow} N'$. By induction hypothesis we have $N' \overset{\beta}{\Longrightarrow} N^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$. Then $(\lambda x.\, N')Q' \overset{\beta}{\Longrightarrow} N^{*\beta}[Q^{*\beta}/x]$ by (4).
   b) If P is not a λ-abstract, then $P'Q' \overset{\beta}{\Longrightarrow} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by eq. (3).

4. The last rule is (4): M is $(\lambda x.\, N)Q$ and M' is $N'[Q'/x]$ for some x, N, Q, N', and Q', with $N \overset{\beta}{\Longrightarrow} N'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we know $N' \overset{\beta}{\Longrightarrow} N^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$. By Lemma cr.4 we have $N'[Q'/x] \overset{\beta}{\Longrightarrow} N^{*\beta}[Q^{*\beta}/x]$, the right-hand side of which is exactly $((\lambda x.\, N)Q)^{*\beta}$. □

Problem cr.3. Complete the proof of Lemma cr.5.

lam:cr:pb: thm:cr

Theorem cr.6. $\overset{\beta}{\Longrightarrow}$ *has the Church–Rosser property.*

Proof. Immediate from Lemma cr.5. □

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Bibliography