cr.1 Parallel $\beta\eta$-reduction

lam:cr:pbe:sec In this section we prove the Church-Rosser property for parallel $\beta\eta$-reduction, the parallel reduction notion corresponding to $\beta\eta$-reduction.

lam:cr:pbe:defn:beredpar **Definition cr.1 (Parallel $\beta\eta$-reduction, $\overset{\beta\eta}{\Longrightarrow}$).** *Parallel $\beta\eta$-reduction* ($\overset{\beta\eta}{\Longrightarrow}$) on terms is inductively defined as follows:

lam:cr:pbe:defn:beredpar1
1. $x \overset{\beta\eta}{\Longrightarrow} x$.

lam:cr:pbe:defn:beredpar2
2. If $N \overset{\beta}{\rightarrow} N'$ then $\lambda x.\, N \overset{\beta\eta}{\Longrightarrow} \lambda x.\, N'$.

lam:cr:pbe:defn:beredpar3
3. If $P \overset{\beta\eta}{\Longrightarrow} P'$ and $Q \overset{\beta\eta}{\Longrightarrow} Q'$ then $PQ \overset{\beta\eta}{\Longrightarrow} P'Q'$.

lam:cr:pbe:defn:beredpar4
4. If $N \overset{\beta\eta}{\Longrightarrow} N'$ and $Q \overset{\beta\eta}{\Longrightarrow} Q'$ then $(\lambda x.\, N)Q \overset{\beta\eta}{\Longrightarrow} N'[Q'/x]$.

lam:cr:pbe:defn:beredpar5
5. If $N \overset{\beta\eta}{\Longrightarrow} N'$ then $\lambda x.\, Nx \overset{\beta\eta}{\Longrightarrow} N'$, provided $x \notin FV(N)$.

lam:cr:pbe:thm:refl **Theorem cr.2.** $M \overset{\beta\eta}{\Longrightarrow} M$.

Proof. Exercise. □

Problem cr.1. Prove Theorem cr.2.

lam:cr:pbe:defn:becd **Definition cr.3 ($\beta\eta$-complete development).** The *$\beta\eta$-complete development* $M^{*\beta\eta}$ of M is defined as follows:

lam:cr:pbe:defn:becd1
$$x^{*\beta\eta} = x \tag{1}$$
lam:cr:pbe:defn:becd2
$$(\lambda x.\, N)^{*\beta\eta} = \lambda x.\, N^{*\beta\eta} \tag{2}$$
lam:cr:pbe:defn:becd3
$$(PQ)^{*\beta\eta} = P^{*\beta\eta}Q^{*\beta\eta} \qquad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{3}$$
lam:cr:pbe:defn:becd4
$$((\lambda x.\, N)Q)^{*\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x] \tag{4}$$
lam:cr:pbe:defn:becd5
$$(\lambda x.\, Nx)^{*\beta\eta} = N^{*\beta\eta} \qquad \text{if } x \notin FV(N) \tag{5}$$

lam:cr:pbe:lem:comp **Lemma cr.4.** *If $M \overset{\beta\eta}{\Longrightarrow} M'$ and $R \overset{\beta\eta}{\Longrightarrow} R'$, then $M[R/y] \overset{\beta\eta}{\Longrightarrow} M'[R'/y]$.*

Proof. By induction on the derivation of $M \overset{\beta\eta}{\Longrightarrow} M'$.

The first four cases are exactly like those in **??**. If the last rule is (5), then M is $\lambda x.\, Nx$, M' is N' for some x and N' where $x \notin FV(N)$, and $N \overset{\beta\eta}{\Longrightarrow} N'$. We want to show that $(\lambda x.\, Nx)[R/y] \overset{\beta\eta}{\Longrightarrow} N'[R'/y]$, i.e., $\lambda x.\, N[R/y]x \overset{\beta\eta}{\Longrightarrow} N'[R'/y]$. It follows by Definition cr.1(5) and the induction hypothesis. □

lam:cr:pbe:lem:cont **Lemma cr.5.** *If $M \overset{\beta\eta}{\Longrightarrow} M'$ then $M' \overset{\beta\eta}{\Longrightarrow} M^{*\beta\eta}$.*

Proof. By induction on the derivation of $M \overset{\beta\eta}{\Longrightarrow} M'$.

The first four cases are like those in **??**. If the last rule is (5), then M is $\lambda x.\, Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \overset{\beta\eta}{\Longrightarrow} N'$. We want to show that $N' \overset{\beta\eta}{\Longrightarrow} (\lambda x.\, Nx)^{*\beta\eta}$, i.e., $N' \overset{\beta\eta}{\Longrightarrow} N^{*\beta\eta}$, which is immediate by induction hypothesis. □

Theorem cr.6. $\overset{\beta\eta}{\Longrightarrow}$ *has the Church-Rosser property.* *lam:cr:pbe:thm:cr*

Proof. Immediate from Lemma cr.5. □

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Bibliography