

## cr.1 Parallel $\beta\eta$ -reduction

`lam:cr:pbe:` In this section we prove the Church-Rosser property for parallel  $\beta\eta$ -reduction,  
`sec` the parallel reduction notion corresponding to  $\beta\eta$ -reduction.

`lam:cr:pbe:` **Definition cr.1 (Parallel  $\beta\eta$ -reduction,  $\xRightarrow{\beta\eta}$ ).** Parallel  $\beta\eta$ -reduction ( $\xRightarrow{\beta\eta}$ )  
`defn:beredpar` on terms is inductively defined as follows:

- `lam:cr:pbe:` 1.  $x \xRightarrow{\beta\eta} x$ .  
`defn:beredpar1`
- `lam:cr:pbe:` 2. If  $N \xrightarrow{\beta} N'$  then  $\lambda x. N \xRightarrow{\beta\eta} \lambda x. N'$ .  
`defn:beredpar2`
- `lam:cr:pbe:` 3. If  $P \xRightarrow{\beta\eta} P'$  and  $Q \xRightarrow{\beta\eta} Q'$  then  $PQ \xRightarrow{\beta\eta} P'Q'$ .  
`defn:beredpar3`
- `lam:cr:pbe:` 4. If  $N \xRightarrow{\beta\eta} N'$  and  $Q \xRightarrow{\beta\eta} Q'$  then  $(\lambda x. N)Q \xRightarrow{\beta\eta} N'[Q'/x]$ .  
`defn:beredpar4`
- `lam:cr:pbe:` 5. If  $N \xRightarrow{\beta\eta} N'$  then  $\lambda x. Nx \xRightarrow{\beta\eta} N'$ , provided  $x \notin FV(N)$ .  
`defn:beredpar5`

`lam:cr:pbe:` **Theorem cr.2.**  $M \xRightarrow{\beta\eta} M$ .  
`thm:refl`

*Proof.* Exercise. □

**Problem cr.1.** Prove [Theorem cr.2](#).

`lam:cr:pbe:` **Definition cr.3 ( $\beta\eta$ -complete development).** The  $\beta\eta$ -complete develop-  
`defn:beed` ment  $M^{*\beta\eta}$  of  $M$  is defined as follows:

$$\text{lam:cr:pbe:} \quad x^{*\beta\eta} = x \tag{1}$$

$$\text{defn:beed1} \quad (\lambda x. N)^{*\beta\eta} = \lambda x. N^{*\beta\eta} \tag{2}$$

$$\text{defn:beed2} \quad (PQ)^{*\beta\eta} = P^{*\beta\eta}Q^{*\beta\eta} \tag{3}$$

$$\text{lam:cr:pbe:} \quad \text{if } P \text{ is not a } \lambda\text{-abstract}$$

$$\text{defn:beed3} \quad ((\lambda x. N)Q)^{*\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x] \tag{4}$$

$$\text{defn:beed4} \quad (\lambda x. Nx)^{*\beta\eta} = N^{*\beta\eta} \tag{5}$$

$$\text{lam:cr:pbe:} \quad \text{if } x \notin FV(N)$$

$$\text{defn:beed5}$$

`lam:cr:pbe:` **Lemma cr.4.** If  $M \xRightarrow{\beta\eta} M'$  and  $R \xRightarrow{\beta\eta} R'$ , then  $M[R/y] \xRightarrow{\beta\eta} M'[R'/y]$ .  
`lem:comp`

*Proof.* By induction on the derivation of  $M \xRightarrow{\beta\eta} M'$ .

The first four cases are exactly like those in ???. If the last rule is (5), then  $M$  is  $\lambda x. Nx$ ,  $M'$  is  $N'$  for some  $x$  and  $N'$  where  $x \notin FV(N)$ , and  $N \xRightarrow{\beta\eta} N'$ . We want to show that  $(\lambda x. Nx)[R/y] \xRightarrow{\beta\eta} N'[R'/y]$ , i.e.,  $\lambda x. N[R/y]x \xRightarrow{\beta\eta} N'[R'/y]$ . It follows by [Definition cr.1\(5\)](#) and the induction hypothesis. □

`lam:cr:pbe:` **Lemma cr.5.** If  $M \xRightarrow{\beta\eta} M'$  then  $M' \xRightarrow{\beta\eta} M^{*\beta\eta}$ .  
`lem:cont`

*Proof.* By induction on the derivation of  $M \xrightarrow{\beta\eta} M'$ .

The first four cases are like those in ???. If the last rule is (5), then  $M$  is  $\lambda x. Nx$  and  $M'$  is  $N'$  for some  $x, N, N'$  where  $x \notin FV(N)$  and  $N \xrightarrow{\beta\eta} N'$ . We want to show that  $N' \xrightarrow{\beta\eta} (\lambda x. Nx)^{*}\beta\eta$ , i.e.,  $N' \xrightarrow{\beta\eta} N^{*\beta\eta}$ , which is immediate by induction hypothesis.  $\square$

**Theorem cr.6.**  $\xrightarrow{\beta\eta}$  has the Church-Rosser property.

*lam:cr:pbe:  
thm:cr*

*Proof.* Immediate from **Lemma cr.5**.  $\square$

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## Bibliography