cr.1 Definition and Properties

lam:cr:dap:sec In this chapter we introduce the concept of Church–Rosser property and some common properties of this property.

Definition cr.1 (Church–Rosser property, CR). A relation $\xrightarrow{X}$ on terms is said to satisfy the *Church–Rosser property* iff, whenever $M \xrightarrow{X} P$ and $M \xrightarrow{X} Q$, then there exists some N such that $P \xrightarrow{X} N$ and $Q \xrightarrow{X} N$.

We can view the lambda calculus as a model of computation in which terms in normal form are "values" and a reducibility relation on terms are the "calculation rules." The Church–Rosser property states is that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there's more than one way to calculate $4 \times (1 + 2) + 3$. It can either be reduced to $4 \times 3 + 3$ (if we first reduce $1 + 2$ to 3) or to $4 \times 1 + 4 \times 2 + 3$ (if we first reduce $4 \times (1 + 2)$ using distributivity). Both of these, however, can be further reduced to $12 + 3$.

If we take $\xrightarrow{X}$ to be β-reduction, we easily see that a consequence of the Church–Rosser property is that if a term has a normal form, then it is unique. For suppose M can be reduced to P and Q, both of which are normal forms. By the Church–Rosser property, there exists some N such that both P and Q reduce to it. Since by assumption P and Q are normal forms, the reduction of P and Q to N can only be the trivial reduction, i.e., P, Q, and N are identical. This justifies our speaking of *the* normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the "final result" of the computation starting with that term. The above corollary means there's only one, if any, final result of a computation, just like there is only one result of computing $4 \times (1 + 2) + 3$, namely 15.

lam:cr:dap:thm:str **Theorem cr.2.** *If a relation $\xrightarrow{X}$ satisfies the Church–Rosser property, and $\xrightarrow{X}\!\!\!\!\rightarrow$ is the smallest transitive relation containing $\xrightarrow{X}$, then $\xrightarrow{X}\!\!\!\!\rightarrow$ satisfies the Church–Rosser property too.*

Proof. Suppose

$$M \xrightarrow{X} P_1 \xrightarrow{X} \dots \xrightarrow{X} P_m \text{ and}$$
$$M \xrightarrow{X} Q_1 \xrightarrow{X} \dots \xrightarrow{X} Q_n.$$

We will prove the theorem by constructing a grid N of terms of height is $m+1$ and width $n + 1$. We use $N_{i,j}$ to denote the term in the i-th row and j-th column.

We construct N in such a way that $N_{i,j} \xrightarrow{X} N_{i+1,j}$ and $N_{i,j} \xrightarrow{X} N_{i,j+1}$. It is defined as follows:

$$\begin{aligned} N_{0,0} &= M \\ N_{i,0} &= P_i && \text{if } 1 \leq i \leq m \\ N_{0,j} &= Q_j && \text{if } 1 \leq j \leq n \end{aligned}$$

and otherwise:

$$N_{i,j} = R$$

where R is a term such that $N_{i-1,j} \xrightarrow{X} R$ and $N_{i,j-1} \xrightarrow{X} R$. By the Church–Rosser property of $\xrightarrow{X}$, such a term always exists.

Now we have $N_{m,0} \xrightarrow{X} \ldots \xrightarrow{X} N_{m,n}$ and $N_{0,n} \xrightarrow{X} \ldots \xrightarrow{X} N_{m,n}$. Note $N_{m,0}$ is P and $N_{0,n}$ is Q. By definition of $\xrightarrow{X}\!\!\!\!\rightarrow$ the theorem follows. $\square$

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Bibliography