

cr.1 β -reduction

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Lemma cr.1. *If $M \xrightarrow{\beta} M'$, then $M \xRightarrow{\beta} M'$.*

Proof. If $M \xrightarrow{\beta} M'$, then M is $(\lambda x. N)Q$, M' is $N[Q/x]$, for some x , N , and Q . Since $N \xRightarrow{\beta} N$ and $Q \xRightarrow{\beta} Q$ by ??, we immediately have $(\lambda x. N)Q \xRightarrow{\beta} N[Q/x]$ by ????? \square

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Lemma cr.2. *If $M \xRightarrow{\beta} M'$, then $M \xrightarrow{\beta} M'$.*

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last rule is ??: Then M and M' are just x , and $x \xrightarrow{\beta} x$.
2. The last rule is ??: M is $\lambda x. N$ and M' is $\lambda x. N'$ for some x , N , N' , where $N \xRightarrow{\beta} N'$. By induction hypothesis we have $N \xrightarrow{\beta} N'$. Then $\lambda x. N \xrightarrow{\beta} \lambda x. N'$ (by the same series of $\xrightarrow{\beta}$ contractions as $N \xrightarrow{\beta} N'$).
3. The last rule is ??: M is PQ and M' is $P'Q'$ for some P , Q , P' , Q' , where $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we have $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. So $PQ \xrightarrow{\beta} P'Q'$ by the reduction sequence $P \xrightarrow{\beta} P'$ followed by the reduction $Q \xrightarrow{\beta} Q'$.
4. The last rule is ??: M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$ for some x , N , M' , Q , Q' , where $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we get $Q \xrightarrow{\beta} Q'$ and $N \xrightarrow{\beta} N'$. So $(\lambda x. N)Q \xrightarrow{\beta} N'[Q'/x]$ by $N \xrightarrow{\beta} N'$ followed by $Q \xrightarrow{\beta} Q'$ and finally contraction of $(\lambda x. N')Q'$ to $N'[Q'/x]$. \square

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Lemma cr.3. *$\xrightarrow{\beta}$ is the smallest transitive relation containing $\xRightarrow{\beta}$.*

Proof. Let \xrightarrow{X} be the smallest transitive relation containing $\xRightarrow{\beta}$.

$\xrightarrow{\beta} \subseteq \xrightarrow{X}$: Suppose $M \xrightarrow{\beta} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. By **Lemma cr.1**, $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. Since \xrightarrow{X} contains $\xRightarrow{\beta}$ and is transitive, $M \xrightarrow{X} M'$.

$\xrightarrow{X} \subseteq \xrightarrow{\beta}$: Suppose $M \xrightarrow{X} M'$, i.e., $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. By **Lemma cr.2**, $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. Since $\xrightarrow{\beta}$ is transitive, $M \xrightarrow{\beta} M'$. \square

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Theorem cr.4. *$\xrightarrow{\beta}$ satisfies the Church-Rosser property.*

Proof. Immediate from ??, ??, and **Lemma cr.3**.

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Bibliography