Lemma cr.1. If $M \xrightarrow{\beta} M'$, then $M \xrightarrow{\beta} M'$.

Proof. If $M \xrightarrow{\beta} M'$, then $M$ is $(\lambda x. N)Q$, $M'$ is $N[Q/x]$, for some $x$, $N$, and $Q$. Since $N \xrightarrow{\beta} N$ and $Q \xrightarrow{\beta} Q$ by ???, we immediately have $(\lambda x. N)Q \xrightarrow{\beta} N[Q/x]$ by ???.

Lemma cr.2. If $M \xrightarrow{\beta} M'$, then $M \xrightarrow{\beta} M'$.

Proof. By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last rule is ???: Then $M$ and $M'$ are just $x$, and $x \xrightarrow{\beta} x$.

2. The last rule is ???: $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$ for some $x$, $N$, $N'$, where $N \xrightarrow{\beta} N'$. By induction hypothesis we have $N \xrightarrow{\beta} N'$. Then $\lambda x. N \xrightarrow{\beta} \lambda x. N'$ (by the same series of $\beta$ contractions as $N \xrightarrow{\beta} N'$).

3. The last rule is ???: $M$ is $PQ$ and $M'$ is $P'Q'$ for some $P$, $Q$, $P'$, $Q'$, where $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis we have $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. So $PQ \xrightarrow{\beta} P'Q'$ by the reduction sequence $P \xrightarrow{\beta} P'$ followed by the reduction $Q \xrightarrow{\beta} Q'$.

4. The last rule is ???: $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$ for some $x$, $N$, $M'$, $Q$, $Q'$, where $N \xrightarrow{\beta} N'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis we get $Q \xrightarrow{\beta} Q'$ and $N \xrightarrow{\beta} N'$. So $(\lambda x. N)Q \xrightarrow{\beta} N'[Q'/x]$ by $N \xrightarrow{\beta} N'$ followed by $Q \xrightarrow{\beta} Q'$ and finally contraction of $(\lambda x. N')Q'$ to $N'[Q'/x]$.

Lemma cr.3. $\xrightarrow{\beta}$ is the smallest transitive relation containing $\xrightarrow{\beta}$.

Proof. Let $\xrightarrow{X}$ be the smallest transitive relation containing $\xrightarrow{\beta}$.

$\xrightarrow{\beta} \subseteq \xrightarrow{X}$: Suppose $M \xrightarrow{\beta} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M'$. By Lemma cr.1, $M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M'$. Since is $X$ contains $\xrightarrow{\beta}$ and is transitive, $M \xrightarrow{X} M'$.

$\xrightarrow{X} \subseteq \xrightarrow{\beta}$: Suppose $M \xrightarrow{X} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M'$. By Lemma cr.2, $M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M'$. Since $\xrightarrow{\beta}$ is transitive, $M \xrightarrow{\beta} M'$.

Theorem cr.4. $\xrightarrow{\beta}$ satisfies the Church–Rosser property.
Proof. Immediate from ??, ??, and Lemma cr.3.

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Bibliography