## cr.1 $\beta$-reduction

lam:cr:b: sec

lam:cr:b: lem:one-par **Lemma cr.1.** *If $M \xrightarrow{\beta} M'$, then $M \overset{\beta}{\Longrightarrow} M'$.*

*Proof.* If $M \xrightarrow{\beta} M'$, then $M$ is $(\lambda x.\, N)Q$, $M'$ is $N[Q/x]$, for some $x$, $N$, and $Q$. Since $N \overset{\beta}{\Longrightarrow} N$ and $Q \overset{\beta}{\Longrightarrow} Q$ by **??**, we immediately have $(\lambda x.\, N)Q \overset{\beta}{\Longrightarrow} N[Q/x]$ by **????**. □

lam:cr:b: lem:par-red **Lemma cr.2.** *If $M \overset{\beta}{\Longrightarrow} M'$, then $M \overset{\beta}{\twoheadrightarrow} M'$.*

*Proof.* By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last rule is **??**: Then $M$ and $M'$ are just $x$, and $x \overset{\beta}{\twoheadrightarrow} x$.

2. The last rule is **??**: $M$ is $\lambda x.\, N$ and $M'$ is $\lambda x.\, N'$ for some $x$, $N$, $N'$, where $N \overset{\beta}{\Longrightarrow} N'$. By induction hypothesis we have $N \overset{\beta}{\twoheadrightarrow} N'$. Then $\lambda x.\, N \overset{\beta}{\twoheadrightarrow} \lambda x.\, N'$ (by the same series of $\xrightarrow{\beta}$ contractions as $N \overset{\beta}{\twoheadrightarrow} N'$).

3. The last rule is **??**: $M$ is $PQ$ and $M'$ is $P'Q'$ for some $P$, $Q$, $P'$, $Q'$, where $P \overset{\beta}{\Longrightarrow} P'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we have $P \overset{\beta}{\twoheadrightarrow} P'$ and $Q \overset{\beta}{\twoheadrightarrow} Q'$. So $PQ \overset{\beta}{\twoheadrightarrow} P'Q'$ by the reduction sequence $P \overset{\beta}{\twoheadrightarrow} P'$ followed by the reduction $Q \overset{\beta}{\twoheadrightarrow} Q'$.

4. The last rule is **??**: $M$ is $(\lambda x.\, N)Q$ and $M'$ is $N'[Q'/x]$ for some $x$, $N$, $M'$, $Q$, $Q'$, where $N \overset{\beta}{\Longrightarrow} N'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we get $Q \overset{\beta}{\twoheadrightarrow} Q'$ and $N \overset{\beta}{\twoheadrightarrow} N'$. So $(\lambda x.\, N)Q \overset{\beta}{\twoheadrightarrow} N'[Q'/x]$ by $N \overset{\beta}{\twoheadrightarrow} N'$ followed by $Q \overset{\beta}{\twoheadrightarrow} Q'$ and finally contraction of $(\lambda x.\, N')Q'$ to $N'[Q'/x]$. □

lam:cr:b: lem:str **Lemma cr.3.** *$\overset{\beta}{\twoheadrightarrow}$ is the smallest transitive relation containing $\overset{\beta}{\Longrightarrow}$.*

*Proof.* Let $\overset{X}{\twoheadrightarrow}$ be the smallest transitive relation containing $\overset{\beta}{\Longrightarrow}$.

$\overset{\beta}{\twoheadrightarrow} \subseteq \overset{X}{\twoheadrightarrow}$: Suppose $M \overset{\beta}{\twoheadrightarrow} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. By Lemma cr.1, $M \equiv M_1 \overset{\beta}{\Longrightarrow} \dots \overset{\beta}{\Longrightarrow} M_k \equiv M'$. Since is $\overset{X}{\twoheadrightarrow}$ contains $\overset{\beta}{\Longrightarrow}$ and is transitive, $M \overset{X}{\twoheadrightarrow} M'$.

$\overset{X}{\twoheadrightarrow} \subseteq \overset{\beta}{\twoheadrightarrow}$: Suppose $M \overset{X}{\twoheadrightarrow} M'$, i.e., $M \equiv M_1 \overset{\beta}{\Longrightarrow} \dots \overset{\beta}{\Longrightarrow} M_k \equiv M'$. By Lemma cr.2, $M \equiv M_1 \overset{\beta}{\twoheadrightarrow} \dots \overset{\beta}{\twoheadrightarrow} M_k \equiv M'$. Since $\overset{\beta}{\twoheadrightarrow}$ is transitive, $M \overset{\beta}{\twoheadrightarrow} M'$. □

lam:cr:b: thm:cr **Theorem cr.4.** *$\overset{\beta}{\twoheadrightarrow}$ satisfies the Church–Rosser property.*

*Proof.* Immediate from **??**, **??**, and Lemma cr.3. □

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## Bibliography