

sc.1 Soundness of Natural Deduction

int:sc:snd:
sec We will now prove soundness of natural deduction with regards to the relational semantics, that is, showing that if a formula is derivable from a set of assumptions then the set of assumptions entails the formula.

int:sc:snd:
thm:soundness **Theorem sc.1 (Soundness).** *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the derivation of φ from Γ .

1. If the derivation consists of just the assumption φ , we have $\varphi \vdash \varphi$, and want to show that $\varphi \vDash \varphi$. Suppose that $\mathfrak{M}, w \Vdash \varphi$. Then trivially $\mathfrak{M}, w \Vdash \varphi$.
2. The derivation ends in \wedge Intro: The derivations of the premises ψ from undischarged assumptions Γ and of χ from undischarged assumptions Δ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \vDash \psi$ and $\Delta \vDash \chi$. We have to show that $\Gamma \cup \Delta \vDash \varphi \wedge \psi$, since the undischarged assumptions of the entire derivation are Γ together with Δ . So suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Then also $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. Similarly, $\mathfrak{M}, w \Vdash \chi$. So $\mathfrak{M}, w \Vdash \psi \wedge \chi$.
3. The derivation ends in \wedge Elim: The derivation of the premise $\psi \wedge \chi$ from undischarged assumptions Γ shows that $\Gamma \vdash \psi \wedge \chi$. By induction hypothesis, $\Gamma \vDash \psi \wedge \chi$. We have to show that $\Gamma \vDash \psi$. So suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi \wedge \chi$, $\mathfrak{M}, w \Vdash \psi \wedge \chi$. Then also $\mathfrak{M}, w \Vdash \psi$. Similarly if \wedge Elim ends in χ , then $\Gamma \vDash \chi$.
4. The derivation ends in \vee Intro: Suppose the premise is ψ , and the undischarged assumptions of the derivation ending in ψ are Γ . Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \vDash \psi$. We have to show that $\Gamma \vDash \psi \vee \chi$. Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. But then also $\mathfrak{M}, w \Vdash \psi \vee \chi$. Similarly, if the premise is χ , we have that $\Gamma \vDash \chi$.
5. The derivation ends in \vee Elim: The derivations ending in the premises are of $\psi \vee \chi$ from undischarged assumptions Γ , of θ from undischarged assumptions $\Delta_1 \cup \{\psi\}$, and of θ from undischarged assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \vdash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vdash \theta$, and $\Delta_2 \cup \{\chi\} \vdash \theta$. By induction hypothesis, $\Gamma \vDash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vDash \theta$, and $\Delta_2 \cup \{\chi\} \vDash \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$.

Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathfrak{M}, w \Vdash \Gamma$ and since $\Gamma \vDash \psi \vee \chi$, $\mathfrak{M}, w \Vdash \psi \vee \chi$. By definition of $\mathfrak{M} \Vdash$, either $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. So we distinguish cases: (a) $\mathfrak{M}, w \Vdash \psi$. Then $\mathfrak{M}, w \Vdash \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \{\psi\} \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. (b) $\mathfrak{M}, w \Vdash \chi$. Then $\mathfrak{M}, w \Vdash \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \{\chi\} \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. So in either case, $\mathfrak{M}, w \Vdash \theta$, as we wanted to show.

6. The **derivation** ends with \rightarrow Intro concluding $\psi \rightarrow \chi$. Then the premise is χ , and the **derivation** ending in the premise has **undischarged** assumptions $\Gamma \cup \{\psi\}$. So we have that $\Gamma \cup \{\psi\} \vdash \chi$, and by induction hypothesis that $\Gamma \cup \{\psi\} \vDash \chi$. We have to show that $\Gamma \vDash \psi \rightarrow \chi$.

Suppose $\mathfrak{M}, w \Vdash \Gamma$. We want to show that for all w' such that Rww' , if $\mathfrak{M}, w' \Vdash \psi$, then $\mathfrak{M}, w' \Vdash \chi$. So assume that Rww' and $\mathfrak{M}, w' \Vdash \psi$. By ??, $\mathfrak{M}, w' \Vdash \Gamma$. Since $\Gamma \cup \{\psi\} \vDash \chi$, $\mathfrak{M}, w' \vDash \chi$, which is what we wanted to show.

7. The **derivation** ends in \rightarrow Elim and conclusion χ . The premises are $\psi \rightarrow \chi$ and ψ , with **derivations** from **undischarged** assumptions Γ, Δ . So we have $\Gamma \vdash \psi \rightarrow \chi$ and $\Delta \vdash \psi$. By inductive hypothesis, $\Gamma \vDash \psi \rightarrow \chi$ and $\Delta \vDash \psi$. We have to show that $\Gamma \cup \Delta \vDash \chi$.

Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \psi \rightarrow \chi$, $\mathfrak{M}, w \vDash \psi \rightarrow \chi$. By definition, this means that for all w' such that Rww' , if $\mathfrak{M}, w' \Vdash \psi$ then $\mathfrak{M}, w' \vDash \chi$. Since R is reflexive, w is among the w' such that Rww' , i.e., we have that if $\mathfrak{M}, w \Vdash \psi$ then $\mathfrak{M}, w \vDash \chi$. Since $\mathfrak{M}, w \Vdash \Delta$ and $\Delta \vDash \psi$, $\mathfrak{M}, w \vDash \psi$. So, $\mathfrak{M}, w \vDash \chi$, as we wanted to show.

8. The **derivation** ends in \perp_I , concluding φ . The premise is \perp and the **undischarged** assumptions of the **derivation** of the premise are Γ . Then $\Gamma \vdash \perp$. By inductive hypothesis, $\Gamma \vDash \perp$. We have to show $\Gamma \vDash \varphi$.

We proceed indirectly. If $\Gamma \not\vDash \varphi$ there is a model \mathfrak{M} and world w such that $\mathfrak{M}, w \Vdash \Gamma$ and $\mathfrak{M}, w \not\vDash \varphi$. Since $\Gamma \vDash \perp$, $\mathfrak{M}, w \vDash \perp$. But that's impossible, since by definition, $\mathfrak{M}, w \not\vDash \perp$. So $\Gamma \vDash \varphi$.

9. The derivation ends in \neg Intro: Exercise.

10. The derivation ends in \neg Elim: Exercise. □

Problem sc.1. Complete the proof of **Theorem sc.1**. For the cases for \neg Intro and \neg Elim, use the definition of $\mathfrak{M}, w \Vdash \neg\varphi$ in ??, i.e., don't treat $\neg\varphi$ as defined by $\varphi \rightarrow \perp$.

Problem sc.2. Show that the following **formulas** are not **derivable** in intuitionistic logic:

1. $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$
2. $(\neg\neg\varphi \rightarrow \varphi) \rightarrow (\varphi \vee \neg\varphi)$
3. $(\varphi \rightarrow \psi \vee \chi) \rightarrow ((\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi))$

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Bibliography