sc.1  Soundness of Natural Deduction

We will now prove soundness of natural deduction with regards to the relational semantics, that is, showing that if a formula is derivable from a set of assumptions then the set of assumptions entails the formula.

Theorem sc.1 (Soundness). If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. The proof is by induction on the derivation of $\varphi$ from $\Gamma$.

1. If the derivation consists of just the assumption $\varphi$, we have $\varphi \vdash \varphi$, and want to show that $\varphi \models \varphi$. Suppose that $\mathcal{M}, w \models \varphi$. Then trivially $\mathcal{M}, w \models \varphi$.

2. The derivation ends in $\land$Intro: The derivations of the premises $\psi$ from undischarged assumptions $\Gamma$ and of $\chi$ from undischarged assumptions $\Delta$ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \models \psi$ and $\Delta \models \chi$. We have to show that $\Gamma \cup \Delta \models \varphi \land \psi$, since the undischarged assumptions of the entire derivation are $\Gamma$ together with $\Delta$. So suppose $\mathcal{M}, w \models \Gamma \cup \Delta$. Then also $\mathcal{M}, w \models \Gamma$. Since $\Gamma \vdash \psi$, $\mathcal{M}, w \models \psi$. Similarly, $\mathcal{M}, w \models \chi$. So $\mathcal{M}, w \models \psi \land \chi$.

3. The derivation ends in $\land$Elim: The derivation of the premise $\psi \land \chi$ from undischarged assumptions $\Gamma$ shows that $\Gamma \vdash \psi \land \chi$. By induction hypothesis, $\Gamma \models \psi \land \chi$. We have to show that $\Gamma \models \psi$. Suppose $\mathcal{M}, w \models \Gamma$. Since $\Gamma \models \psi \land \chi$, $\mathcal{M}, w \models \psi \land \chi$. Then also $\mathcal{M}, w \models \psi$. Similarly if $\land$Elim ends in $\chi$, then $\Gamma \models \chi$.

4. The derivation ends in $\lor$Intro: Suppose the premise is $\psi$, and the undischarged assumptions of the derivation ending in $\psi$ are $\Gamma$. Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \models \psi$. We have to show that $\Gamma \models \psi \lor \chi$. Suppose $\mathcal{M}, w \models \Gamma$. Since $\Gamma \models \psi$, $\mathcal{M}, w \models \psi$. But then also $\mathcal{M}, w \models \psi \lor \chi$. Similarly, if the premise is $\chi$, we have that $\Gamma \models \chi$.

5. The derivation ends in $\lor$Elim: The derivations ending in the premises are of $\psi \lor \chi$ from undischarged assumptions $\Gamma$, of $\theta$ from undischarged assumptions $\Delta_1 \cup \{\psi\}$, and of $\theta$ from undischarged assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \models \psi \lor \chi$, $\Delta_1 \cup \{\psi\} \models \theta$, and $\Delta_2 \cup \{\chi\} \models \theta$. By induction hypothesis, $\Gamma \models \psi \lor \chi$, $\Delta_1 \cup \{\psi\} \models \theta$, and $\Delta_2 \cup \{\chi\} \models \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \models \theta$.

Suppose $\mathcal{M}, w \models \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathcal{M}, w \models \Gamma$ and since $\Gamma \models \psi \lor \chi$, $\mathcal{M}, w \models \psi \lor \chi$. By definition of $\models$, either $\mathcal{M}, w \models \psi$ or $\mathcal{M}, w \models \chi$. So we distinguish cases: (a) $\mathcal{M} \models \psi[w]$. Then $\mathcal{M}, w \models \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \{\psi\} \models \theta$, we have $\mathcal{M}, w \models \theta$. (b) $\mathcal{M}, w \models \chi$. Then $\mathcal{M}, w \models \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \chi \models \theta$, we have $\mathcal{M}, w \models \theta$. So in either case, $\mathcal{M}, w \models \theta$, as we wanted to show.
6. The derivation ends with →Intro concluding \( \psi \to \chi \). Then the premise is \( \chi \), and the derivation ending in the premise has undischarged assumptions \( \Gamma \cup \{ \psi \} \). So we have that \( \Gamma \cup \{ \psi \} \vdash \chi \), and by induction hypothesis that \( \Gamma \cup \{ \psi \} \models \chi \). We have to show that \( \Gamma \models \psi \to \chi \).

Suppose \( \mathcal{M}, w \models \Gamma \). We want to show that for all \( w' \) such that \( Rww' \), if \( \mathcal{M}, w' \models \psi \), then \( \mathcal{M}, w' \models \chi \). So assume that \( Rww' \) and \( \mathcal{M}, w' \models \psi \). By \ref{??}, \( \mathcal{M}, w' \models \Gamma \). Since \( \Gamma \cup \{ \psi \} \models \chi \), which is what we wanted to show.

7. The derivation ends in →Elim and conclusion \( \chi \). The premises are \( \psi \to \chi \) and \( \psi \), with derivations from undischarged assumptions \( \Gamma, \Delta \). So we have \( \Gamma \vdash \psi \to \chi \) and \( \Delta \vdash \psi \). By inductive hypothesis, \( \Gamma \models \psi \to \chi \) and \( \Delta \models \psi \). We have to show that \( \Gamma \cup \Delta \models \chi \).

Suppose \( \mathcal{M}, w \models \Gamma \cup \Delta \). Since \( \mathcal{M}, w \models \Gamma \) and \( \Gamma \models \psi \to \chi \), \( \mathcal{M}, w \models \psi \to \chi \). By definition, this means that for all \( w' \) such that \( Rww' \), if \( \mathcal{M}, w' \models \psi \) then \( \mathcal{M}, w' \models \chi \). Since \( R \) is reflexive, \( w \) is among the \( w' \) such that \( Rww' \), i.e., we have that if \( \mathcal{M}, w \models \psi \) then \( \mathcal{M}, w \models \chi \). Since \( \mathcal{M}, w \models \Delta \) and \( \Delta \models \psi \), \( \mathcal{M}, w \models \psi \). So, \( \mathcal{M}, w \models \chi \), as we wanted to show.

8. The derivation ends in \( \bot \)I, concluding \( \varphi \). The premise is \( \bot \) and the undischarged assumptions of the derivation of the premise are \( \Gamma \). Then \( \Gamma \vdash \bot \). By inductive hypothesis, \( \Gamma \models \bot \). We have to show that \( \Gamma \models \varphi \).

We proceed indirectly. If \( \Gamma \not\models \varphi \) there is a model \( \mathcal{M} \) and world \( w \) such that \( \mathcal{M}, w \models \Gamma \) and \( \mathcal{M}, w \not\models \varphi \). Since \( \Gamma \models \bot \), \( \mathcal{M}, w \models \bot \). But that’s impossible, since by definition, \( \mathcal{M}, w \not\models \bot \). So \( \Gamma \not\models \varphi \).

9. The derivation ends in \( \neg \)Intro: Exercise.

10. The derivation ends in \( \neg \)Elim: Exercise.

Problem sc.1. Complete the proof of Theorem sc.1. For the cases for \( \neg \)Intro and \( \neg \)Elim, use the definition of \( \mathcal{M}, w \models \neg \varphi \) in \ref{??}, i.e., don’t treat \( \neg \varphi \) as defined by \( \varphi \to \bot \).

Problem sc.2. Show that the following formulas are not derivable in intuitionistic logic:

1. \((\varphi \to \psi) \lor (\psi \to \varphi)\)
2. \((\neg \neg \varphi \to \varphi) \to ((\varphi \lor \neg \varphi))\)
3. \((\varphi \to \psi \lor \chi) \to ((\varphi \to \psi) \lor (\varphi \to \chi))\)