

## Chapter udf

# Soundness and Completeness

This chapter collects soundness and completeness results for propositional intuitionistic logic. It needs an introduction. The completeness proof makes use of facts about provability that should be stated and proved explicitly somewhere.

### sc.1 Soundness of Axiomatic Derivations

int:sc:sax:  
sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

int:sc:sar:  
thm:soundness

**Theorem sc.1 (Soundness).** *If  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ .*

*Proof.* We prove that if  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ . The proof is by induction on the number  $n$  of formulas in the derivation of  $\varphi$  from  $\Gamma$ . We show that if  $\varphi_1, \dots, \varphi_n = \varphi$  is a derivation from  $\Gamma$ , then  $\Gamma \vDash \varphi_n$ . Note that if  $\varphi_1, \dots, \varphi_n$  is a derivation, so is  $\varphi_1, \dots, \varphi_k$  for any  $k < n$ .

There are no derivations of length 0, so for  $n = 0$  the claim holds vacuously. So the claim holds for all derivations of length  $< n$ . We distinguish cases according to the justification of  $\varphi_n$ .

1.  $\varphi_n$  is an axiom. All axioms are valid, so  $\Gamma \vDash \varphi_n$  for any  $\Gamma$ .
2.  $\varphi_n \in \Gamma$ . Then for any  $\mathfrak{M}$  and  $w$ , if  $\mathfrak{M}, w \Vdash \Gamma$ , obviously  $\mathfrak{M} \Vdash \Gamma \varphi_n[w]$ , i.e.,  $\Gamma \vDash \varphi$ .
3.  $\varphi_n$  follows by MP from  $\varphi_i$  and  $\varphi_j \equiv \varphi_i \rightarrow \varphi_n$ .  $\varphi_1, \dots, \varphi_i$  and  $\varphi_1, \dots, \varphi_j$  are derivations from  $\Gamma$ , so by inductive hypothesis,  $\Gamma \vDash \varphi_i$  and  $\Gamma \vDash \varphi_i \rightarrow \varphi_n$ .

Suppose  $\mathfrak{M}, w \Vdash \Gamma$ . Since  $\mathfrak{M}, w \Vdash \Gamma$  and  $\Gamma \vDash \varphi_i \rightarrow \varphi_n$ ,  $\mathfrak{M}, w \Vdash \varphi_i \rightarrow \varphi_n$ . By definition, this means that for all  $w'$  such that  $Rww'$ , if  $\mathfrak{M}, w' \Vdash \varphi_i$  then  $\mathfrak{M}, w' \Vdash \varphi_n$ . Since  $R$  is reflexive,  $w$  is among the  $w'$  such that  $Rww'$ , i.e., we have that if  $\mathfrak{M}, w \Vdash \varphi_i$  then  $\mathfrak{M}, w \Vdash \varphi_n$ . Since  $\Gamma \vDash \varphi_i$ ,  $\mathfrak{M}, w \Vdash \varphi_i$ . So,  $\mathfrak{M}, w \Vdash \varphi_n$ , as we wanted to show.  $\square$

## sc.2 Soundness of Natural Deduction

We will now prove soundness of natural deduction with regards to the relational semantics, that is, showing that if a formula is derivable from a set of assumptions then the set of assumptions entails the formula.

int:sc:snd:  
sec

**Theorem sc.2 (Soundness).** *If  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ .*

int:sc:snd:  
thm:soundness

*Proof.* We prove that if  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ . The proof is by induction on the derivation of  $\varphi$  from  $\Gamma$ .

1. If the derivation consists of just the assumption  $\varphi$ , we have  $\varphi \vdash \varphi$ , and want to show that  $\varphi \vDash \varphi$ . Suppose that  $\mathfrak{M}, w \Vdash \varphi$ . Then trivially  $\mathfrak{M}, w \Vdash \varphi$ .
2. The derivation ends in  $\wedge$ Intro: The derivations of the premises  $\psi$  from undischarged assumptions  $\Gamma$  and of  $\chi$  from undischarged assumptions  $\Delta$  show that  $\Gamma \vdash \psi$  and  $\Delta \vdash \chi$ . By induction hypothesis we have that  $\Gamma \vDash \psi$  and  $\Delta \vDash \chi$ . We have to show that  $\Gamma \cup \Delta \vDash \varphi \wedge \psi$ , since the undischarged assumptions of the entire derivation are  $\Gamma$  together with  $\Delta$ . So suppose  $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$ . Then also  $\mathfrak{M}, w \Vdash \Gamma$ . Since  $\Gamma \vDash \psi$ ,  $\mathfrak{M}, w \Vdash \psi$ . Similarly,  $\mathfrak{M}, w \Vdash \chi$ . So  $\mathfrak{M}, w \Vdash \psi \wedge \chi$ .
3. The derivation ends in  $\wedge$ Elim: The derivation of the premise  $\psi \wedge \chi$  from undischarged assumptions  $\Gamma$  shows that  $\Gamma \vdash \psi \wedge \chi$ . By induction hypothesis,  $\Gamma \vDash \psi \wedge \chi$ . We have to show that  $\Gamma \vDash \psi$ . So suppose  $\mathfrak{M}, w \Vdash \Gamma$ . Since  $\Gamma \vDash \psi \wedge \chi$ ,  $\mathfrak{M}, w \Vdash \psi \wedge \chi$ . Then also  $\mathfrak{M}, w \Vdash \psi$ . Similarly if  $\wedge$ Elim ends in  $\chi$ , then  $\Gamma \vDash \chi$ .
4. The derivation ends in  $\vee$ Intro: Suppose the premise is  $\psi$ , and the undischarged assumptions of the derivation ending in  $\psi$  are  $\Gamma$ . Then we have  $\Gamma \vdash \psi$  and by inductive hypothesis,  $\Gamma \vDash \psi$ . We have to show that  $\Gamma \vDash \psi \vee \chi$ . Suppose  $\mathfrak{M}, w \Vdash \Gamma$ . Since  $\Gamma \vDash \psi$ ,  $\mathfrak{M}, w \Vdash \psi$ . But then also  $\mathfrak{M}, w \Vdash \psi \vee \chi$ . Similarly, if the premise is  $\chi$ , we have that  $\Gamma \vDash \chi$ .
5. The derivation ends in  $\vee$ Elim: The derivations ending in the premises are of  $\psi \vee \chi$  from undischarged assumptions  $\Gamma$ , of  $\theta$  from undischarged assumptions  $\Delta_1 \cup \{\psi\}$ , and of  $\theta$  from undischarged assumptions  $\Delta_2 \cup \{\chi\}$ . So we have  $\Gamma \vdash \psi \vee \chi$ ,  $\Delta_1 \cup \{\psi\} \vdash \theta$ , and  $\Delta_2 \cup \{\chi\} \vdash \theta$ . By induction hypothesis,  $\Gamma \vDash \psi \vee \chi$ ,  $\Delta_1 \cup \{\psi\} \vDash \theta$ , and  $\Delta_2 \cup \{\chi\} \vDash \theta$ . We have to prove that  $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$ .

Suppose  $\mathfrak{M}, w \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$ . Then  $\mathfrak{M}, w \Vdash \Gamma$  and since  $\Gamma \vDash \psi \vee \chi$ ,  $\mathfrak{M}, w \Vdash \psi \vee \chi$ . By definition of  $\mathfrak{M} \Vdash$ , either  $\mathfrak{M}, w \Vdash \psi$  or  $\mathfrak{M}, w \Vdash \chi$ . So we distinguish cases: (a)  $\mathfrak{M} \Vdash \psi[w]$ . Then  $\mathfrak{M}, w \Vdash \Delta_1 \cup \{\psi\}$ . Since  $\Delta_1 \cup \psi \vDash \theta$ , we have  $\mathfrak{M}, w \Vdash \theta$ . (b)  $\mathfrak{M}, w \Vdash \chi$ . Then  $\mathfrak{M}, w \Vdash \Delta_2 \cup \{\chi\}$ . Since  $\Delta_2 \cup \chi \vDash \theta$ , we have  $\mathfrak{M}, w \Vdash \theta$ . So in either case,  $\mathfrak{M}, w \Vdash \theta$ , as we wanted to show.

6. The **derivation** ends with  $\rightarrow$ Intro concluding  $\psi \rightarrow \chi$ . Then the premise is  $\chi$ , and the **derivation** ending in the premise has **undischarged** assumptions  $\Gamma \cup \{\psi\}$ . So we have that  $\Gamma \cup \{\psi\} \vdash \chi$ , and by induction hypothesis that  $\Gamma \cup \{\psi\} \vDash \chi$ . We have to show that  $\Gamma \vDash \psi \rightarrow \chi$ .

Suppose  $\mathfrak{M}, w \Vdash \Gamma$ . We want to show that for all  $w'$  such that  $Rww'$ , if  $\mathfrak{M}, w' \Vdash \psi$ , then  $\mathfrak{M}, w' \Vdash \chi$ . So assume that  $Rww'$  and  $\mathfrak{M}, w' \Vdash \psi$ . By ??,  $\mathfrak{M}, w' \Vdash \Gamma$ . Since  $\Gamma \cup \{\psi\} \vDash \chi$ ,  $\mathfrak{M}, w' \Vdash \chi$ , which is what we wanted to show.

7. The **derivation** ends in  $\rightarrow$ Elim and conclusion  $\chi$ . The premises are  $\psi \rightarrow \chi$  and  $\psi$ , with **derivations** from **undischarged** assumptions  $\Gamma, \Delta$ . So we have  $\Gamma \vdash \psi \rightarrow \chi$  and  $\Delta \vdash \psi$ . By inductive hypothesis,  $\Gamma \vDash \psi \rightarrow \chi$  and  $\Delta \vDash \psi$ . We have to show that  $\Gamma \cup \Delta \vDash \chi$ .

Suppose  $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$ . Since  $\mathfrak{M}, w \Vdash \Gamma$  and  $\Gamma \vDash \psi \rightarrow \chi$ ,  $\mathfrak{M}, w \Vdash \psi \rightarrow \chi$ . By definition, this means that for all  $w'$  such that  $Rww'$ , if  $\mathfrak{M}, w' \Vdash \psi$  then  $\mathfrak{M}, w' \Vdash \chi$ . Since  $R$  is reflexive,  $w$  is among the  $w'$  such that  $Rww'$ , i.e., we have that if  $\mathfrak{M}, w \Vdash \psi$  then  $\mathfrak{M}, w \Vdash \chi$ . Since  $\mathfrak{M}, w \Vdash \Delta$  and  $\Delta \vDash \psi$ ,  $\mathfrak{M}, w \Vdash \psi$ . So,  $\mathfrak{M}, w \Vdash \chi$ , as we wanted to show.

8. The **derivation** ends in  $\perp_I$ , concluding  $\varphi$ . The premise is  $\perp$  and the **undischarged** assumptions of the **derivation** of the premise are  $\Gamma$ . Then  $\Gamma \vdash \perp$ . By inductive hypothesis,  $\Gamma \vDash \perp$ . We have to show  $\Gamma \vDash \varphi$ .

We proceed indirectly. If  $\Gamma \not\vDash \varphi$  there is a model  $\mathfrak{M}$  and world  $w$  such that  $\mathfrak{M}, w \Vdash \Gamma$  and  $\mathfrak{M}, w \not\vDash \varphi$ . Since  $\Gamma \vDash \perp$ ,  $\mathfrak{M}, w \Vdash \perp$ . But that's impossible, since by definition,  $\mathfrak{M}, w \not\vDash \perp$ . So  $\Gamma \vDash \varphi$ .

9. The derivation ends in  $\neg$ Intro: Exercise.

10. The derivation ends in  $\neg$ Elim: Exercise. □

**Problem sc.1.** Complete the proof of **Theorem sc.2**. For the cases for  $\neg$ Intro and  $\neg$ Elim, use the definition of  $\mathfrak{M}, w \Vdash \neg\varphi$  in ??, i.e., don't treat  $\neg\varphi$  as defined by  $\varphi \rightarrow \perp$ .

**Problem sc.2.** Show that the following **formulas** are not **derivable** in intuitionistic logic:

1.  $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$
2.  $(\neg\neg\varphi \rightarrow \varphi) \rightarrow (\varphi \vee \neg\varphi)$
3.  $(\varphi \rightarrow \psi \vee \chi) \rightarrow ((\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi))$

### sc.3 Lindenbaum's Lemma

The completeness theorem for intuitionistic logic is proved by assuming  $\Gamma \not\vdash \varphi$  and constructing a model  $\mathfrak{M} \Vdash \Gamma$  and  $\mathfrak{M} \not\vdash \varphi$ .

In classical logic the relation of **derivability** can be reduced to the notion of consistency since a **formula**  $\varphi$  is **derivable** from a set of **formulas** iff the set together with the negation of  $\varphi$  is inconsistent. This is not possible in intuitionistic logic. In intuitionistic logic, if  $\neg\varphi$  is inconsistent, we only get that  $\vdash \neg\neg\varphi$ . Since  $\neg\neg\varphi \rightarrow \varphi$  does not hold intuitionistically in general, we cannot conclude that  $\vdash \varphi$ .

Thus, when constructing the model  $\mathfrak{M}$ , we will need to keep track of the non-**derivability** of the **formula**  $\varphi$  and thus we will not be able to use a complete set  $\Gamma^* \supseteq \Gamma$  to build the model  $\mathfrak{M}$ , as in every complete set  $\Gamma^*$ , we have  $\Gamma^* \vdash \varphi \vee \neg\varphi$ .

Instead of using a complete set  $\Gamma^*$ , we will use the notion of a prime set of formulas:

**Definition sc.3.** A set of **formulas**  $\Gamma$  is *prime* iff

1.  $\Gamma$  is consistent, i.e.,  $\Gamma \not\vdash \perp$ ;
2. if  $\Gamma \vdash \varphi$  then  $\varphi \in \Gamma$ ; and
3. if  $\varphi \vee \psi \in \Gamma$  then  $\varphi \in \Gamma$  or  $\psi \in \Gamma$ .

**Lemma sc.4 (Lindenbaum's Lemma).** *If  $\Gamma \not\vdash \varphi$ , there is a  $\Gamma^* \supseteq \Gamma$  such that  $\Gamma^*$  is prime and  $\Gamma^* \not\vdash \varphi$ .*

*Proof.* Let  $\psi_1 \vee \chi_1, \psi_2 \vee \chi_2, \dots$ , be an enumeration of all **formulas** of the form  $\psi \vee \chi$ . We'll define an increasing sequence of sets of **formulas**  $\Gamma_n$ , where each  $\Gamma_{n+1}$  is defined as  $\Gamma_n$  together with one new **formula**.  $\Gamma^*$  will be the union of all  $\Gamma_n$ . The new **formulas** are selected so as to ensure that  $\Gamma^*$  is prime and still  $\Gamma^* \not\vdash \varphi$ . This means that at each step we should find the first disjunction  $\psi_i \vee \chi_i$  such that:

1.  $\Gamma_n \vdash \psi_i \vee \chi_i$
2.  $\psi_i \notin \Gamma_n$  and  $\chi_i \notin \Gamma_n$

We add to  $\Gamma_n$  either  $\psi_i$  if  $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$ , or  $\chi_i$  otherwise. We'll have to show that this works. For now, let's define  $i(n)$  as the least  $i$  such that (1) and (2) hold.

Define  $\Gamma_0 = \Gamma$  and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \not\vdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If  $i(n)$  is undefined, i.e., whenever  $\Gamma_n \vdash \psi \vee \chi$ , either  $\psi \in \Gamma_n$  or  $\chi \in \Gamma_n$ , we let  $\Gamma_{n+1} = \Gamma_n$ . Now let  $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all  $n$ ,  $\Gamma_n \not\vdash \varphi$ . We proceed by induction on  $n$ . For  $n = 0$  the claim holds by the hypothesis of the theorem, i.e.,  $\Gamma \not\vdash \varphi$ . If  $n > 0$ , we have to show that if  $\Gamma_n \not\vdash \varphi$  then  $\Gamma_{n+1} \not\vdash \varphi$ . If  $i(n)$  is undefined,  $\Gamma_{n+1} = \Gamma_n$  and there is nothing to prove. So suppose  $i(n)$  is defined. For simplicity, let  $i = i(n)$ .

We'll prove the contrapositive of the claim. Suppose  $\Gamma_{n+1} \vdash \varphi$ . By construction,  $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$  if  $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$ , or else  $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$ . It clearly can't be the first, since then  $\Gamma_{n+1} \not\vdash \varphi$ . Hence,  $\Gamma_n \cup \{\psi_i\} \vdash \varphi$  and  $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$ . By definition of  $i(n)$ , we have that  $\Gamma_n \vdash \psi_i \vee \chi_i$ . We have  $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ . We also have  $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$ . Hence,  $\Gamma_n \vdash \varphi$ , which is what we wanted to show.

If  $\Gamma^* \vdash \varphi$ , there would be some finite subset  $\Gamma' \subseteq \Gamma^*$  such that  $\Gamma' \vdash \varphi$ . Each  $\theta \in \Gamma'$  must be in  $\Gamma_i$  for some  $i$ . Let  $n$  be the largest of these. Since  $\Gamma_i \subseteq \Gamma_n$  if  $i \leq n$ ,  $\Gamma' \subseteq \Gamma_n$ . But then  $\Gamma_n \vdash \varphi$ , contrary to our proof above that  $\Gamma_n \not\vdash \varphi$ .

Lastly, we show that  $\Gamma^*$  is prime, i.e., satisfies conditions (1), (2), and (3) of [Definition sc.3](#).

First,  $\Gamma^* \not\vdash \varphi$ , so  $\Gamma^*$  is consistent, so (1) holds.

We now show that if  $\Gamma^* \vdash \psi \vee \chi$ , then either  $\psi \in \Gamma^*$  or  $\chi \in \Gamma^*$ . This proves (3), since if  $\psi \vee \chi \in \Gamma^*$  then also  $\Gamma^* \vdash \psi \vee \chi$ . So assume  $\Gamma^* \vdash \psi \vee \chi$  but  $\psi \notin \Gamma^*$  and  $\chi \notin \Gamma^*$ . Since  $\Gamma^* \vdash \psi \vee \chi$ ,  $\Gamma_n \vdash \psi \vee \chi$  for some  $n$ .  $\psi \vee \chi$  appears on the enumeration of all disjunctions, say, as  $\psi_j \vee \chi_j$ .  $\psi_j \vee \chi_j$  satisfies the properties in the definition of  $i(n)$ , namely we have  $\Gamma_n \vdash \psi_j \vee \chi_j$ , while  $\psi_j \notin \Gamma_n$  and  $\chi_j \notin \Gamma_n$ . At each stage, at least one fewer disjunction  $\psi_i \vee \chi_i$  satisfies the conditions (since at each stage we add either  $\psi_i$  or  $\chi_i$ ), so at some stage  $m$  we will have  $j = i(m)$ . But then either  $\psi \in \Gamma_{m+1}$  or  $\chi \in \Gamma_{m+1}$ , contrary to the assumption that  $\psi \notin \Gamma^*$  and  $\chi \notin \Gamma^*$ .

Now suppose  $\Gamma^* \vdash \psi$ . Then  $\Gamma^* \vdash \psi \vee \psi$ . But we've just proved that if  $\Gamma^* \vdash \psi \vee \psi$  then  $\psi \in \Gamma^*$ . Hence,  $\Gamma^*$  satisfies (2) of [Definition sc.3](#).  $\square$

**Problem sc.3.** Show that if  $\Gamma \not\vdash \perp$  then  $\Gamma$  is consistent in classical logic, i.e., there is a valuation making all formulas in  $\Gamma$  true.

## sc.4 The Canonical Model

int:sc:mod:  
sec The worlds in our model will be finite sequences  $\sigma$  of natural numbers, i.e.,  $\sigma \in \mathbb{N}^*$ . Note that  $\mathbb{N}^*$  is inductively defined by:

1.  $\Lambda \in \mathbb{N}^*$ .
2. If  $\sigma \in \mathbb{N}^*$  and  $n \in \mathbb{N}$ , then  $\sigma.n \in \mathbb{N}^*$  (where  $\sigma.n$  is  $\sigma \frown \langle n \rangle$  and  $\sigma \frown \sigma'$  is the concatenation of  $\sigma$  and  $\sigma'$ ).
3. Nothing else is in  $\mathbb{N}^*$ .

So we can use  $\mathbb{N}^*$  to give inductive definitions.

Let  $\langle \psi_1, \chi_1 \rangle, \langle \psi_2, \chi_2 \rangle, \dots$  be an enumeration of all pairs of [formulas](#). Given a set of [formulas](#)  $\Delta$ , define  $\Delta(\sigma)$  by induction as follows:

1.  $\Delta(\Lambda) = \Delta$
2.  $\Delta(\sigma.n) = \begin{cases} (\Delta(\sigma) \cup \{\psi_n\})^* & \text{if } \Delta(\sigma) \cup \{\psi_n\} \not\vdash \chi_n \\ \Delta(\sigma) & \text{otherwise} \end{cases}$

Here by  $(\Delta(\sigma) \cup \{\psi_n\})^*$  we mean the prime set of formulas which exists by [Lemma sc.4](#) applied to the set  $\Delta(\sigma) \cup \{\psi_n\}$  and the formula  $\chi_n$ . Note that by this definition, if  $\Delta(\sigma) \cup \{\psi_n\} \not\vdash \chi_n$ , then  $\Delta(\sigma.n) \vdash \psi_n$  and  $\Delta(\sigma.n) \not\vdash \chi_n$ . Note also that  $\Delta(\sigma) \subseteq \Delta(\sigma.n)$  for any  $n$ . If  $\Delta$  is prime, then  $\Delta(\sigma)$  is prime for all  $\sigma$ .

**Definition sc.5.** Suppose  $\Delta$  is prime. Then the *canonical model*  $\mathfrak{M}(\Delta)$  for  $\Delta$  is defined by:

[int:sc:mod:](#)  
[defn:canonical-model](#)

1.  $W = \mathbb{N}^*$ , the set of finite sequences of natural numbers.
2.  $R$  is the partial order according to which  $R\sigma\sigma'$  iff  $\sigma$  is an initial segment of  $\sigma'$  (i.e.,  $\sigma' = \sigma \frown \sigma''$  for some sequence  $\sigma''$ ).
3.  $V(p) = \{\sigma : p \in \Delta(\sigma)\}$ .

It is easy to verify that  $R$  is indeed a partial order. Also, the monotonicity condition on  $V$  is satisfied. Since  $\Delta(\sigma) \subseteq \Delta(\sigma.n)$  we get  $\Delta(\sigma) \subseteq \Delta(\sigma')$  whenever  $R\sigma\sigma'$  by induction on  $\sigma$ .

## sc.5 The Truth Lemma

**Lemma sc.6.** *If  $\Delta$  is prime, then  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\Delta(\sigma) \vdash \varphi$ .*

[int:sc:tru:](#)  
[sec](#)  
[int:sc:tru:](#)  
[lem:truth](#)

*Proof.* By induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ : Since  $\Delta(\sigma)$  is prime, it is consistent, so  $\Delta(\sigma) \not\vdash \varphi$ . By definition,  $\mathfrak{M}(\Delta), \sigma \not\vdash \varphi$ .
2.  $\varphi \equiv p$ : By definition of  $\Vdash$ ,  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\sigma \in V(p)$ , i.e.,  $\Delta(\sigma) \vdash \varphi$ .
3.  $\varphi \equiv \neg\psi$ : exercise.
4.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  and  $\mathfrak{M}(\Delta), \sigma \Vdash \chi$ . By induction hypothesis,  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  iff  $\Delta(\sigma) \vdash \psi$ , and similarly for  $\chi$ . But  $\Delta(\sigma) \vdash \psi$  and  $\Delta(\sigma) \vdash \chi$  iff  $\Delta(\sigma) \vdash \varphi$ .
5.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  or  $\mathfrak{M}(\Delta), \sigma \Vdash \chi$ . By induction hypothesis, this holds iff  $\Delta(\sigma) \vdash \psi$  or  $\Delta(\sigma) \vdash \chi$ . We have to show that this in turn holds iff  $\Delta(\sigma) \vdash \varphi$ . The left-to-right direction is clear. The right-to-left direction follows since  $\Delta(\sigma)$  is prime.

6.  $\varphi \equiv \psi \rightarrow \chi$ : First the contrapositive of the left-to-right direction: Assume  $\Delta(\sigma) \not\vdash \psi \rightarrow \chi$ . Then also  $\Delta(\sigma) \cup \{\psi\} \not\vdash \chi$ . Since  $\langle \psi, \chi \rangle$  is  $\langle \psi_n, \chi_n \rangle$  for some  $n$ , we have  $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$ , and  $\Delta(\sigma.n) \vdash \psi$  but  $\Delta(\sigma.n) \not\vdash \chi$ . By inductive hypothesis,  $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$  and  $\mathfrak{M}(\Delta), \sigma.n \not\vdash \chi$ . Since  $R\sigma(\sigma.n)$ , this means that  $\mathfrak{M}(\Delta), \sigma \not\vdash \varphi$ .

Now assume  $\Delta(\sigma) \vdash \psi \rightarrow \chi$ , and let  $R\sigma\sigma'$ . Since  $\Delta(\sigma) \subseteq \Delta(\sigma')$ , we have: if  $\Delta(\sigma') \vdash \psi$ , then  $\Delta(\sigma') \vdash \chi$ . In other words, for every  $\sigma'$  such that  $R\sigma\sigma'$ , either  $\Delta(\sigma') \not\vdash \psi$  or  $\Delta(\sigma') \vdash \chi$ . By induction hypothesis, this means that whenever  $R\sigma\sigma'$ , either  $\mathfrak{M}(\Delta), \sigma' \not\vdash \psi$  or  $\mathfrak{M}(\Delta), \sigma' \Vdash \chi$ , i.e.,  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ .  $\square$

## sc.6 The Completeness Theorem

int:sc:epl:

sec

int:sc:epl:

thm:completeness

**Theorem sc.7.** *If  $\Gamma \models \varphi$  then  $\Gamma \vdash \varphi$ .*

*Proof.* We prove the contrapositive: Suppose  $\Gamma \not\vdash \varphi$ . Then by **Lemma sc.4**, there is a prime set  $\Gamma^* \supseteq \Gamma$  such that  $\Gamma^* \not\vdash \varphi$ . Consider the canonical model  $\mathfrak{M}(\Gamma^*)$  for  $\Gamma^*$  as defined in **Definition sc.5**. For any  $\psi \in \Gamma$ ,  $\Gamma^* \vdash \psi$ . Note that  $\Gamma^*(\Delta) = \Gamma^*$ . By the Truth Lemma (**Lemma sc.6**), we have  $\mathfrak{M}(\Gamma^*), \Delta \Vdash \psi$  for all  $\psi \in \Gamma$  and  $\mathfrak{M}(\Gamma^*), \Delta \not\vdash \varphi$ . This shows that  $\Gamma \not\vdash \varphi$ .  $\square$

**Problem sc.4.** Show that if  $\varphi$  only contains **propositional variables**,  $\vee$ , and  $\wedge$ , then  $\not\vdash \varphi$ . Use this to conclude that  $\rightarrow$  is not definable in intuitionistic logic from  $\vee$  and  $\wedge$ .

**Problem sc.5.** By using the completeness theorem prove that if  $\vdash \varphi \vee \psi$  then  $\vdash \varphi$  or  $\vdash \psi$ . (Hint: Assume  $\mathfrak{M}_1 \not\vdash \varphi$  and  $\mathfrak{M}_2 \not\vdash \psi$  and construct a new model  $\mathfrak{M}$  such that  $\mathfrak{M} \not\vdash \varphi \vee \psi$ .)

**Problem sc.6.** Show that if  $\mathfrak{M}$  is a relational model using a linear order then  $\mathfrak{M} \Vdash (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ .

## sc.7 Decidability

int:sc:dec:

sec

Observe that the proof of the completeness theorem gives us for every  $\Gamma \not\vdash \varphi$  a model with an infinite number of worlds witnessing the fact that  $\Gamma \not\vdash \varphi$ . The following proposition shows that to prove  $\models \varphi$  it is enough to prove that  $\mathfrak{M} \Vdash \varphi$  for all finite models (i.e., models with a finite set of worlds).

int:sc:dec:

thm:decidability

**Theorem sc.8.** *If  $\not\vdash \varphi$  then there is a finite model  $\mathfrak{M}' \not\vdash \varphi$ .*

*Proof.* Assume  $\mathfrak{M} = \langle W, R, V \rangle$  is such that  $\mathfrak{M} \not\vdash \varphi$  and  $P$  is the set of **propositional variables** occurring in  $\varphi$ . Define  $\mathfrak{M}' = \langle W', R', V' \rangle$  by letting  $W' = \{[w] : w \in W\}$  where  $[w] = \{p \in P : w \in V(p)\}$ ,  $R'$  be the subset

relation, and  $V'(p) = \{[w] : p \in [w]\}$ . It should be clear that  $W'$  is a finite set and that  $\mathfrak{M}'$  is a relational model.

It can be shown, by induction on  $\varphi$ , that

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', [w] \Vdash \varphi$$

for all formulas  $\varphi$  with only propositional variables from  $P$ . This is left as an exercise for the reader.  $\square$

**Problem sc.7.** Finish the proof of [Theorem sc.8](#) by showing that  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}', [w] \Vdash \varphi$  for all formulas  $\varphi$  with only propositional variables from  $P$ .

From [Theorem sc.8](#) it follows that there is an algorithm to decide whether  $\models \varphi$ .

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# Bibliography