The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

**Theorem sc.1 (Soundness).** If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

**Proof.** We prove that if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. The proof is by induction on the number $n$ of formulas in the derivation of $\varphi$ from $\Gamma$. We show that if $\varphi_1, \ldots, \varphi_n = \varphi$ is a derivation from $\Gamma$, then $\Gamma \models \varphi_n$. Note that if $\varphi_1, \ldots, \varphi_n$ is a derivation, so is $\varphi_1, \ldots, \varphi_k$ for any $k < n$.

There are no derivations of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivations of length $< n$. We distinguish cases according to the justification of $\varphi_n$.

1. $\varphi_n$ is an axiom. All axioms are valid, so $\Gamma \models \varphi_n$ for any $\Gamma$.

2. $\varphi_n \in \Gamma$. Then for any $M$ and $w$, if $M, w \models \Gamma$, obviously $M \models \Gamma \varphi_n[w]$, i.e., $\Gamma \models \varphi$.

3. $\varphi_n$ follows by $\text{mp}$ from $\varphi_i$ and $\varphi_j \equiv \varphi_i \rightarrow \varphi_n$. $\varphi_1, \ldots, \varphi_i$ and $\varphi_1, \ldots, \varphi_j$ are derivations from $\Gamma$, so by inductive hypothesis, $\Gamma \models \varphi_i$ and $\Gamma \models \varphi_i \rightarrow \varphi_n$.

   Suppose $M, w \models \Gamma$. Since $M, w \models \Gamma$ and $\Gamma \models \varphi_i \rightarrow \varphi_n$, $M, w \models \varphi_i \rightarrow \varphi_n$. By definition, this means that for all $w'$ such that $Rww'$, if $M, w' \models \varphi_i$, then $M, w' \models \varphi_n$. Since $R$ is reflexive, $w$ is among the $w'$ such that $Rww'$, i.e., we have that if $M, w \models \varphi_i$ then $M, w \models \varphi_n$. Since $\Gamma \models \varphi_i$, $M, w \models \varphi_i$. So, $M, w \models \varphi_n$, as we wanted to show.

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**Bibliography**