sc.1 Soundness of Axiomatic Derivations

int:sc:sax:sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

int:sc:sax:thm:soundness

Theorem sc.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the number n of formulas in the derivation of φ from Γ. We show that if φ_1, …, $\varphi_n = \varphi$ is a derivation from Γ, then $\Gamma \vDash \varphi_n$. Note that if φ_1, …, φ_n is a derivation, so is φ_1, …, φ_k for any $k < n$.

There are no derivations of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivations of length $< n$. We distinguish cases according to the justification of φ_n.

1. φ_n is an axiom. All axioms are valid, so $\Gamma \vDash \varphi_n$ for any Γ.

2. $\varphi_n \in \Gamma$. Then for any $\mathfrak{M}$ and w, if $\mathfrak{M}, w \Vdash \Gamma$, obviously $\mathfrak{M} \Vdash \Gamma\varphi_n[w]$, i.e., $\Gamma \vDash \varphi$.

3. φ_n follows by MP from φ_i and $\varphi_j \equiv \varphi_i \to \varphi_n$. φ_1, …, φ_i and φ_1, …, φ_j are derivations from Γ, so by inductive hypothesis, $\Gamma \vDash \varphi_i$ and $\Gamma \vDash \varphi_i \to \varphi_n$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi_i \to \varphi_n$, $\mathfrak{M}, w \Vdash \varphi_i \to \varphi_n$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \varphi_i$ then $\mathfrak{M}, w' \Vdash \varphi_n$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \varphi_i$ then $\mathfrak{M}, w \Vdash \varphi_n$. Since $\Gamma \vDash \varphi_i$, $\mathfrak{M}, w \Vdash \varphi_i$. So, $\mathfrak{M}, w \Vdash \varphi_n$, as we wanted to show. □

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Bibliography