sc.1  Lindenbaum’s Lemma

The completeness theorem for intuitionistic logic is proved by assuming $\Gamma \not\vdash \varphi$ and constructing a model $\mathcal{M} \models \Gamma$ and $\mathcal{M} \not\models \varphi$.

In classical logic the relation of derivability can be reduced to the notion of consistency since a formula $\varphi$ is derivable from a set of formulas if and only if the set together with the negation of $\varphi$ is inconsistent. This is not possible in intuitionistic logic. In intuitionistic logic, if $\neg \varphi$ is inconsistent, we only get that $\Gamma \vdash \neg \neg \varphi$. Since $\neg \neg \varphi \rightarrow \varphi$ does not hold intuitionistically in general, we cannot conclude that $\Gamma \vdash \varphi$.

Thus, when constructing the model $\mathcal{M}$, we will need to keep track of the non-derivability of the formula $\varphi$ and thus we will not be able to use a complete set $\Gamma^* \supseteq \Gamma$ to build the model $\mathcal{M}$, as in every complete set $\Gamma^*$, we have $\Gamma^* \vdash \varphi \lor \neg \varphi$.

Instead of using a complete set $\Gamma^*$, we will use the notion of a prime set of formulas:

**Definition sc.1.** A set of formulas $\Gamma$ is prime iff

1. $\Gamma$ is consistent, i.e., $\Gamma \not\vdash \bot$;
2. if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$; and
3. if $\varphi \lor \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

**Lemma sc.2 (Lindenbaum’s Lemma).** If $\Gamma \not\vdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that $\Gamma^*$ is prime and $\Gamma^* \not\vdash \varphi$.

**Proof.** Let $\psi_1 \lor \chi_1, \psi_2 \lor \chi_2, \ldots$, be an enumeration of all formulas of the form $\psi \lor \chi$. We’ll define an increasing sequence of sets of formulas $\Gamma_n$, where each $\Gamma_{n+1}$ is defined as $\Gamma_n$ together with one new formula. $\Gamma^*$ will be the union of all $\Gamma_n$. The new formulas are selected so as to ensure that $\Gamma^*$ is prime and still $\Gamma^* \not\vdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \lor \chi_i$ such that:

1. $\Gamma_n \vdash \psi_i \lor \chi_i$
2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to $\Gamma_n$ either $\psi_i$ if $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$, or $\chi_i$ otherwise. We’ll have to show that this works. For now, let’s define $i(n)$ as the least $i$ such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$
\Gamma_{n+1} = \begin{cases} 
\Gamma_n \cup \{\psi_i(n)\} & \text{if } \Gamma_n \cup \{\psi_i(n)\} \not\vdash \varphi \\
\Gamma_n \cup \{\chi_i(n)\} & \text{otherwise}
\end{cases}
$$

If $i(n)$ is undefined, i.e., whenever $\Gamma_n \vdash \psi \lor \chi$, either $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$.
First we show that for all \( n \), \( \Gamma_n \not\vdash \varphi \). We proceed by induction on \( n \). For \( n = 0 \) the claim holds by the hypothesis of the theorem, i.e., \( \Gamma \not\vdash \varphi \). If \( n > 0 \), we have to show that if \( \Gamma_n \not\vdash \varphi \) then \( \Gamma_{n+1} \not\vdash \varphi \). If \( i(n) \) is undefined, \( \Gamma_{n+1} = \Gamma_n \) and there is nothing to prove. So suppose \( i(n) \) is defined. For simplicity, let \( i = i(n) \).

We'll prove the contrapositive of the claim. Suppose \( \Gamma_{n+1} \vdash \varphi \). By construction, \( \Gamma_{n+1} = \Gamma_n \cup \{ \psi \} \) if \( \Gamma_n \cup \{ \psi \} \not\vdash \varphi \), or else \( \Gamma_{n+1} = \Gamma_n \cup \{ \chi \} \). It clearly can't be the first, since then \( \Gamma_{n+1} \not\vdash \varphi \). Hence, \( \Gamma_n \cup \{ \psi \} \vdash \varphi \) and \( \Gamma_{n+1} = \Gamma_n \cup \{ \chi \} \). By definition of \( i(n) \), we have that \( \Gamma_n \vdash \psi \lor \chi \). We have \( \Gamma_n \cup \{ \psi \} \vdash \varphi \). We also have \( \Gamma_{n+1} = \Gamma_n \cup \{ \chi \} \vdash \varphi \). Hence, \( \Gamma_n \vdash \varphi \), which is what we wanted to show.

If \( \Gamma^* \vdash \varphi \), there would be some finite subset \( \Gamma' \subseteq \Gamma^* \) such that \( \Gamma' \vdash \varphi \).

Each \( \theta \in \Gamma' \) must be in \( \Gamma_i \) for some \( i \). Let \( n \) be the largest of these. Since \( \Gamma_i \subseteq \Gamma_n \) if \( i \leq n \), \( \Gamma' \subseteq \Gamma_n \). But then \( \Gamma_n \vdash \varphi \), contrary to our proof above that \( \Gamma_n \not\vdash \varphi \).

Lastly, we show that \( \Gamma^* \) is prime, i.e., satisfies conditions (1), (2), and (3) of Definition sc.1.

First, \( \Gamma^* \not\vdash \varphi \), so \( \Gamma^* \) is consistent, so (1) holds.

We now show that if \( \Gamma^* \vdash \psi \lor \chi \), then either \( \psi \in \Gamma^* \) or \( \chi \in \Gamma^* \). This proves (3), since if \( \psi \lor \chi \in \Gamma^* \) then also \( \Gamma^* \vdash \psi \lor \chi \). So assume \( \Gamma^* \vdash \psi \lor \chi \) but \( \psi \notin \Gamma^* \) and \( \chi \notin \Gamma^* \). Since \( \Gamma^* \vdash \psi \lor \chi \), \( \Gamma_n \vdash \psi \lor \chi \) for some \( n \). \( \psi \lor \chi \) appears on the enumeration of all disjunctions, say, as \( \psi_j \lor \chi_j \). \( \psi_j \lor \chi_j \) satisfies the properties in the definition of \( i(n) \), namely we have \( \Gamma_n \vdash \psi_j \lor \chi_j \), while \( \psi_j \notin \Gamma_n \) and \( \chi_j \notin \Gamma_n \). At each stage, at least one fewer disjunction \( \psi_i \lor \chi_i \) satisfies the conditions (since at each stage we add either \( \psi_i \) or \( \chi_i \)), so at some stage \( m \) we will have \( j = i(m) \). But then either \( \psi \in \Gamma_{m+1} \) or \( \chi \in \Gamma_{m+1} \), contrary to the assumption that \( \psi \notin \Gamma^* \) and \( \chi \notin \Gamma^* \).

Now suppose \( \Gamma^* \vdash \psi \). Then \( \Gamma^* \vdash \psi \lor \psi \). But we’ve just proved that if \( \Gamma^* \vdash \psi \lor \psi \) then \( \psi \in \Gamma^* \). Hence, \( \Gamma^* \) satisfies (2) of Definition sc.1.

\textbf{Problem sc.1.} Show that if \( \Gamma \not\vdash \bot \) then \( \Gamma \) is consistent in classical logic, i.e., there is a valuation making all formulas in \( \Gamma \) true.

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