sc.1 The Completeness Theorem

int:sc:cpl:sec
int:sc:cpl:thm:completeness

Theorem sc.1. *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. We prove the contrapositive: Suppose $\Gamma \nvdash \varphi$. Then by **??**, there is a prime set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \nvdash \varphi$. Consider the canonical model $\mathfrak{M}(\Gamma^*)$ for Γ^* as defined in **??**. For any $\psi \in \Gamma$, $\Gamma^* \vdash \psi$. Note that $\Gamma^*(\Lambda) = \Gamma^*$. By the Truth Lemma (**??**), we have $\mathfrak{M}(\Gamma^*), \Lambda \Vdash \psi$ for all $\psi \in \Gamma$ and $\mathfrak{M}(\Gamma^*), \Lambda \nVdash \varphi$. This shows that $\Gamma \nvDash \varphi$. □

Problem sc.1. Show that if φ only contains propositional variables, $\lor$, and $\land$, then $\nvDash \varphi$. Use this to conclude that $\to$ is not definable in intuitionistic logic from $\lor$ and $\land$.

Problem sc.2. By using the completeness theorem prove that if $\vdash \varphi \lor \psi$ then $\vdash \varphi$ or $\vdash \psi$. (Hint: Assume $\mathfrak{M}_1 \nVdash \varphi$ and $\mathfrak{M}_2 \nVdash \psi$ and construct a new model $\mathfrak{M}$ such that $\mathfrak{M} \nVdash \varphi \lor \psi$.)

Problem sc.3. Show that if $\mathfrak{M}$ is a relational model using a linear order then $\mathfrak{M} \Vdash (\varphi \to \psi) \lor (\psi \to \varphi)$.

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Bibliography