### Topological Semantics

Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology.

**Definition sem.1.** Let $X$ be a set. A **topology on $X$** is a set $\mathcal{O} \subseteq \mathcal{P}(X)$ that satisfies the properties below. The elements of $\mathcal{O}$ are called the **open sets** of the topology. The set $X$ together with $\mathcal{O}$ is called a **topological space**.

1. The empty set and the entire space are open: $\emptyset, X \in \mathcal{O}$.
2. Open sets are closed under finite intersections: if $U, V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$.
3. Open sets are closed under arbitrary unions: if $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\{U_i : i \in I\} \in \mathcal{O}$.

We may write $X$ for a topology if the collection of open sets can be inferred from the context; note that, still, only after $X$ is endowed with open sets can it be called a topology.

**Definition sem.2.** A **topological model** of intuitionistic propositional logic is a triple $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$ where $\mathcal{O}$ is a topology on $X$ and $V$ is a function assigning an open set in $\mathcal{O}$ to each propositional variable.

Given a topological model $\mathfrak{X}$, we can define $[\varphi]_{\mathfrak{X}}$ inductively as follows:

1. $[\bot]_{\mathfrak{X}} = \emptyset$
2. $[p]_{\mathfrak{X}} = V(p)$
3. $[\varphi \land \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$
4. $[\varphi \lor \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$
5. $[\varphi \rightarrow \psi]_{\mathfrak{X}} = \text{Int}(\{U \setminus [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}\})$

Here, $\text{Int}(V)$ is the function that maps a set $V \subseteq X$ to its **interior**, that is, the union of all open sets it contains. In other words,

$$\text{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus, $[\varphi]_{\mathfrak{X}}$ is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the elements, or “points,” of $X$ are points at which statements can be evaluated. The set of all points where $\varphi$ is true is the proposition expressed by $\varphi$. Not every set of points is a potential proposition; only the elements of $\mathcal{O}$ are. $\varphi \vdash \psi$ iff $\psi$ is true at every point at which $\varphi$ is true, i.e., $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, for all $X$. The absurd statement $\bot$ is never true, so $[\bot]_{\mathfrak{X}} = \emptyset$. 

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How must the propositions expressed by $\psi \land \chi$, $\psi \lor \chi$, and $\psi \rightarrow \chi$ be related to those expressed by $\psi$ and $\chi$ for the intuitionistically valid laws to hold, i.e., so that $\varphi \vdash \psi$ iff $[\varphi]_X \subseteq [\psi]_X$? We require $\perp \vdash \varphi$ for any $\varphi$, which is satisfied because $\emptyset \subseteq U$ for all $U$. Since $\psi \land \chi \vdash \psi$, we require that $[\psi \land \chi]_X \subseteq [\psi]_X$, and similarly $[\psi \land \chi]_X \subseteq [\chi]_X$. The largest set satisfying $W \subseteq U$ and $W \subseteq V$ is $U \cap V$. Conversely, $\psi \vdash \psi \lor \chi$ and $\chi \vdash \psi \lor \chi$, and so we require that $[\psi]_X \subseteq [\psi \lor \chi]_X$ and $[\chi]_X \subseteq [\psi \lor \chi]_X$. The smallest set $W$ such that $U \subseteq W$ and $V \subseteq W$ is $U \cup V$.

The definition for $\rightarrow$ is tricky: $\varphi \rightarrow \psi$ expresses the weakest proposition that, combined with $\varphi$, entails $\psi$. That $\varphi \rightarrow \psi$ combined with $\varphi$ entails $\psi$ is clear from $(\varphi \rightarrow \psi) \land \varphi \vdash \psi$. So $[\varphi \rightarrow \psi]_X$ should be the greatest open set such that $[\varphi \rightarrow \psi]_X \cap [\varphi]_X \subseteq [\psi]_X$, leading to our definition.

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**Bibliography**