

pty.1 Sequent Natural Deduction

int:pty:snd: Let us write $\Gamma \Rightarrow \varphi$ if there is a natural deduction **derivation** with Γ as **undis-**
sec **charged** assumptions and φ as conclusion; or $\Rightarrow \varphi$ if Γ is empty.

We write $\Gamma, \varphi_1, \dots, \varphi_n$ for $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$, and Γ, Δ for $\Gamma \cup \Delta$.

Observe that when we have $\Gamma \Rightarrow \varphi \wedge \psi$, meaning we have a **derivation** with Γ as **undischarged** assumptions and $\varphi \wedge \psi$ as end-**formula**, then by applying \wedge Elim at the bottom, we can get a **derivation** with the same **undischarged** assumptions and φ as conclusion. In other words, if $\Gamma \Rightarrow \varphi \wedge \psi$, then $\Gamma \Rightarrow \varphi$.

$$\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi} \wedge\text{Elim} \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi} \wedge\text{Elim}$$

The label \wedge Elim hints at the relation with the rule of the same name in natural deduction.

Likewise, suppose we have $\Gamma, \varphi \Rightarrow \psi$, meaning we have a **derivation** with **undischarged** assumptions Γ, φ and end-**formula** ψ . If we apply the \rightarrow Intro rule, we have a **derivation** with Γ as **undischarged** assumptions and $\varphi \rightarrow \psi$ as the end-**formula**, i.e., $\Gamma \Rightarrow \varphi \rightarrow \psi$. Note how this has made the **discharge** of assumptions more explicit.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{Intro}$$

We can draw conclusions from other rules in the same fashion, which is spelled out as follows:

$$\begin{array}{c} \frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \wedge \psi} \wedge\text{Intro} \\ \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi} \wedge\text{Elim}_1 \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi} \wedge\text{Elim}_2 \\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \vee\text{Intro}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} \vee\text{Intro}_2 \\ \frac{\Gamma \Rightarrow \varphi \vee \psi \quad \Delta, \varphi \Rightarrow \chi \quad \Delta', \psi \Rightarrow \chi}{\Gamma, \Delta, \Delta' \Rightarrow \chi} \vee\text{Elim} \\ \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi} \rightarrow\text{Intro} \qquad \frac{\Delta \Rightarrow \varphi \rightarrow \psi \quad \Gamma \Rightarrow \varphi}{\Gamma, \Delta \Rightarrow \psi} \rightarrow\text{Elim} \\ \frac{\Gamma \Rightarrow \perp}{\Gamma \Rightarrow \varphi} \perp_I \end{array}$$

Any assumption by itself is a **derivation** of φ from φ , i.e., we always have $\varphi \Rightarrow \varphi$.

$$\overline{\varphi \Rightarrow \varphi}$$

Together, these rules can be taken as a calculus about what natural deduction **derivations** exist. They can also be taken as a notational variant of natural deduction, in which each step records not only the **formula derived** but also the **undischarged** assumptions from which it was **derived**.

$$\frac{\frac{\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee (\varphi \rightarrow \perp)}}{\varphi, \psi \rightarrow \Rightarrow \perp} \quad \psi \Rightarrow \psi}{\frac{(\psi \Rightarrow \varphi \rightarrow \perp)}{(\psi \Rightarrow \varphi \vee (\varphi \rightarrow \perp))} \quad (\psi \Rightarrow \psi)}{\frac{(\psi \Rightarrow \perp)}{\Rightarrow \psi \rightarrow \perp}}$$

where ψ is short for $(\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp$.

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Bibliography