pty.1 Sequent Natural Deduction

int:pty:snd:sec Let us write $\Gamma \Rightarrow \varphi$ if there is a natural deduction derivation with Γ as undischarged assumptions and φ as conclusion; or $\Rightarrow \varphi$ if Γ is empty.

We write $\Gamma, \varphi_1, \dots, \varphi_n$ for $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$, and Γ, Δ for $\Gamma \cup \Delta$.

Observe that when we have $\Gamma \Rightarrow \varphi \land \varphi$, meaning we have a derivation with Γ as undischarged assumptions and $\varphi \land \varphi$ as end-formula, then by applying $\land$Elim at the bottom, we can get a derivation with the same undischarged assumptions and φ as conclusion. In other words, if $\Gamma \Rightarrow \varphi \land \psi$, then $\Gamma \Rightarrow \varphi$.

$$\frac{\Gamma \Rightarrow \varphi \land \psi}{\Gamma \Rightarrow \varphi}\land\text{Elim} \qquad \frac{\Gamma \Rightarrow \varphi \land \psi}{\Gamma \Rightarrow \psi}\land\text{Elim}$$

The label $\land$Elim hints at the relation with the rule of the same name in natural deduction.

Likewise, suppose we have $\Gamma, \varphi \Rightarrow \psi$, meaning we have a derivation with undischarged assumptions Γ, φ and end-formula ψ. If we apply the $\rightarrow$Intro rule, we have a derivation with Γ as undischarged assumptions and $\varphi \rightarrow \psi$ as the end-formula, i.e., $\Gamma \Rightarrow \varphi \rightarrow \psi$. Note how this has made the discharge of assumptions more explicit.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}\rightarrow\text{Intro}$$

We can draw conclusions from other rules in the same fashion, which is spelled out as follows:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \land \psi}\land\text{Intro}$$

$$\frac{\Gamma \Rightarrow \varphi \land \psi}{\Gamma \Rightarrow \varphi}\land\text{Elim}_1 \qquad \frac{\Gamma \Rightarrow \varphi \land \psi}{\Gamma \Rightarrow \psi}\land\text{Elim}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \lor \psi}\lor\text{Intro}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \lor \psi}\lor\text{Intro}_2$$

$$\frac{\Gamma \Rightarrow \varphi \lor \psi \qquad \Delta, \varphi \Rightarrow \chi \qquad \Delta', \psi \Rightarrow \chi}{\Gamma, \Delta, \Delta' \Rightarrow \chi}\lor\text{Elim}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \rightarrow \psi}\rightarrow\text{Intro} \qquad \frac{\Delta \Rightarrow \varphi \rightarrow \psi \qquad \Gamma \Rightarrow \varphi}{\Gamma, \Delta \Rightarrow \psi}\rightarrow\text{Elim}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\bot_I$$

Any assumption by itself is a derivation of φ from φ, i.e., we always have $\varphi \Rightarrow \varphi$.

$$\frac{}{\varphi \Rightarrow \varphi}$$

Together, these rules can be taken as a calculus about what natural deduction derivations exist. They can also be taken as a notational variant of natural deduction, in which each step records not only the formula derived but also the undischarged assumptions from which it was derived.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \vee (\varphi \to \bot)} \qquad \psi \Rightarrow \psi}{\varphi, \psi \to \;\Rightarrow\; \bot}}{(\psi \Rightarrow \varphi \to \bot}}{(\psi \Rightarrow \varphi \vee (\varphi \to \bot)} \qquad (\psi \Rightarrow \psi}{(\psi \Rightarrow \bot}}{\Rightarrow \psi \to \bot}$$

where ψ is short for $(\varphi \vee (\varphi \to \bot)) \to \bot$.

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Bibliography