We give the definition of proof terms, and then establish its relation with natural deduction derivations.

**Definition pty.1 (Proof terms).** Proof terms are inductively generated by the following rules:

1. A single variable $x$ is a proof term.
2. If $P$ and $Q$ are proof terms, then $PQ$ is also a proof term.
3. If $x$ is a variable, $\varphi$ is a formula, and $N$ is a proof term, then $\lambda x^\varphi . N$ is also a proof term.
4. If $P$ and $Q$ are proof terms, then $\langle P, Q \rangle$ is a proof term.
5. If $M$ is a proof term, then $p_i(M)$ is also a proof term, where $i$ is 1 or 2.
6. If $M$ is a proof term, and $\varphi$ is a formula, then $\in_i^\varphi (M)$ is a proof term, where $i$ is 1 or 2.
7. If $M, N_1, N_2$ is proof terms, and $x_1, x_2$ are variables, then $\text{case}(M, x_1 . N_1, x_2 . N_2)$ is a proof term.
8. If $M$ is a proof term and $\varphi$ is a formula, then $\text{contr}_{\varphi}(M)$ is proof term.

Each of the above rules corresponds to an inference rule in natural deduction. Thus we can inductively assign proof terms to the formulas in a derivation. To make this assignment unique, we must distinguish between the two versions of $\land$ Elim and of $\lor$ Intro. For instance, the proof terms assigned to the conclusion of $\lor$ Intro must carry the information whether $\varphi \lor \psi$ is inferred from $\varphi$ or from $\psi$. Suppose $M$ is the term assigned to $\varphi$ from which $\varphi \lor \psi$ is inferred. Then the proof term assigned to $\varphi \lor \psi$ is $\in_1^\varphi (M)$. If we instead infer $\psi \lor \varphi$ then the proof term assigned is $\in_2^\varphi (M)$.

The term $\lambda x^\varphi . N$ is assigned to the conclusion of $\rightarrow$ Intro. The $\varphi$ represents the assumption being discharged; only have we included it can we infer the formula of $\lambda x^\varphi . N$ based on the formula of $N$.

**Definition pty.2 (Typing context).** A *typing context* is a mapping from variables to formulas. We will call it simply the “context” if there is no confusion. We write a context $\Gamma$ as a set of pairs $\langle x, \varphi \rangle$.

A pair $\Gamma \Rightarrow M$ where $M$ is a proof term represents a derivation of a formula with context $\Gamma$.

**Definition pty.3 (Typing pair).** A *typing pair* is a pair $\langle \Gamma, M \rangle$, where $\Gamma$ is a typing context and $M$ is a proof term.
Since in general terms only make sense with specific contexts, we will speak simply of “terms” from now on instead of “typing pair”; and it will be apparent when we are talking about the literal term $M$.

Photo Credits

Bibliography