pty.1 Proof Terms

int:pty:ter:sec We give the definition of proof terms, and then establish its relation with natural deduction derivations.

Definition pty.1 (Proof terms). Proof terms are inductively generated by the following rules:

1. A single variable x is a proof term.
2. If P and Q are proof terms, then PQ is also a proof term.
3. If x is a variable, φ is a formula, and N is a proof term, then $\lambda x^{\varphi}.\, N$ is also a proof term.
4. If P and Q are proof terms, then $\langle P, Q\rangle$ is a proof term.
5. If M is a proof term, then $\mathrm{p}_i(M)$ is also a proof term, where i is 1 or 2.
6. If M is a proof term, and φ is a formula, then $\mathrm{in}_i^{\varphi}(M)$ is a proof term, where i is 1 or 2.
7. If M, N_1, N_2 is proof terms, and x_1, x_2 are variables, then $\mathrm{case}(M, x_1.N_1, x_2.N_2)$ is a proof term.
8. If M is a proof term and φ is a formula, then $\mathrm{contr}_{\varphi}(M)$ is proof term.

Each of the above rules corresponds to an inference rule in natural deduction. Thus we can inductively assign proof terms to the formulas in a derivation. To make this assignment unique, we must distinguish between the two versions of $\wedge$Elim and of $\vee$Intro. For instance, the proof terms assigned to the conclusion of $\vee$Intro must carry the information whether $\varphi \vee \psi$ is inferred from φ or from ψ. Suppose M is the term assigned to $\varphi$from which $\varphi \vee \psi$ is inferred. Then the proof term assigned to $\varphi \vee \psi$ is $\mathrm{in}_1^{\varphi}(M)$. If we instead infer $\psi \vee \varphi$ then the proof term assigned is $\mathrm{in}_2^{\varphi}(M)$.

The term $\lambda x^{\varphi}.\, N$ is assigned to the conclusion of $\to$Intro. The φ represents the assumption being discharged; only have we included it can we infer the formula of $\lambda x^{\varphi}.\, N$ based on the formula of N.

Definition pty.2 (Typing context). A *typing context* is a mapping from variables to formulas. We will call it simply the "context" if there is no confusion. We write a context Γ as a set of pairs $\langle x, \varphi\rangle$.

A pair $\Gamma \Rightarrow M$ where M is a proof term represents a derivation of a formula with context Γ.

Definition pty.3 (Typing pair). A *typing pair* is a pair $\langle \Gamma, M\rangle$, where Γ is a typing context and M is a proof term.

Since in general terms only make sense with specific contexts, we will speak simply of "terms" from now on instead of "typing pair"; and it will be apparent when we are talking about the literal term M.

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Bibliography