

## pty.1 Normalization

int:pty:nor:  
sec In this section we prove that, via some reduction order, any deduction can be reduced to a normal deduction, which is called the *normalization property*. We will make use of the propositions-as-types correspondence: we show that every proof term can be reduced to a normal form; normalization for natural deduction *derivations* then follows.

Firstly we define some functions that measure the complexity of terms. The *length*  $\text{len}(\varphi)$  of a *formulas* is defined by

$$\begin{aligned}\text{len}(p) &= 0 \\ \text{len}(\varphi \wedge \psi) &= \text{len}(\varphi) + \text{len}(\psi) + 1 \\ \text{len}(\varphi \vee \psi) &= \text{len}(\varphi) + \text{len}(\psi) + 1 \\ \text{len}(\varphi \rightarrow \psi) &= \text{len}(\varphi) + \text{len}(\psi) + 1.\end{aligned}$$

The complexity of a redex  $M$  is measured by its *cut rank*  $\text{cr}(M)$ :

$$\begin{aligned}\text{cr}((\lambda x^\varphi. N^\psi)Q) &= \text{len}(\varphi) + \text{len}(\psi) + 1 \\ \text{cr}(\text{p}_i(\langle M^\varphi, N^\psi \rangle)) &= \text{len}(\varphi) + \text{len}(\psi) + 1 \\ \text{cr}(\text{case}(\text{in}_i^{\varphi_i}(M^{\varphi_i}), x_1^{\varphi_1}.N_1^X, x_2^{\varphi_2}.N_2^X)) &= \text{len}(\varphi) + \text{len}(\psi) + 1\end{aligned}$$

The complexity of a proof term is measured by the most complex redex in it, and 0 if it is normal:

$$\text{mr}(M) = \max\{\text{cr}(N) \mid N \text{ is a sub term of } M \text{ and is redex}\}$$

int:pty:nor:  
lem:subst **Lemma pty.1.** *If  $M[N^\varphi/x^\varphi]$  is a redex and  $M \not\equiv x$ , then one of the following cases holds:*

1.  $M$  is itself a redex, or
2.  $M$  is of the form  $\text{p}_i(x)$ , and  $N$  is of the form  $\langle P_1, P_2 \rangle$
3.  $M$  is of the form  $\text{case}(i, x_1.P_1, x_2.P_2)$ , and  $N$  is of the form  $\text{in}_i(Q)$
4.  $M$  is of the form  $xQ$ , and  $N$  is of the form  $\lambda x. P$

*In the first case,  $\text{cr}(M[N/x]) = \text{cr}(M)$ ; in the other cases,  $\text{cr}(M[N/x]) = \text{len}(\varphi)$ .*

*Proof.* Proof by induction on  $M$ .

1. If  $M$  is a single variable  $y$  and  $y \not\equiv x$ , then  $y[N/x]$  is  $y$ , hence not a redex.
2. If  $M$  is of the form  $\langle N_1, N_2 \rangle$ , or  $\lambda x. N$ , or  $\text{in}_i^\varphi(N)$ , then  $M[N^\varphi/x^\varphi]$  is also of that form, and so is not a redex.
3. If  $M$  is of the form  $\text{p}_i(P)$ , we consider two cases.

- a) If  $P$  is of the form  $\langle P_1, P_2 \rangle$ , then  $M \equiv p_i(\langle P_1, P_2 \rangle)$  is a redex, and clearly

$$M[N/x] \equiv p_i(\langle P_1[N/x], P_2[N/x] \rangle)$$

is also a redex. The cut ranks are equal.

- b) If  $P$  is a single variable, it must be  $x$  to make the substitution a redex, and  $N$  must be of the form  $\langle P_1, P_2 \rangle$ . Now consider

$$M[N/x] \equiv p_i(x)[\langle P_1, P_2 \rangle/x],$$

which is  $p_i(\langle P_1, P_2 \rangle)$ . Its cut rank is equal to  $\text{cr}(x)$ , which is  $\text{len}(\varphi)$ .

The cases of  $\text{case}(N, x_1.N_1, x_2.N_2)$  and  $PQ$  are similar.  $\square$

**Lemma pty.2.** *If  $M$  contracts to  $M'$ , and  $\text{cr}(M) > \text{cr}(N)$  for all proper redex sub-terms  $N$  of  $M$ , then  $\text{cr}(M) > \text{mr}(M')$ .*

*Proof.* Proof by cases.

1. If  $M$  is of the form  $p_i(\langle M_1, M_2 \rangle)$ , then  $M'$  is  $M_i$ ; since any sub-term of  $M_i$  is also proper sub-term of  $M$ , the claim holds.
2. If  $M$  is of the form  $(\lambda x^\varphi. N)Q^\varphi$ , then  $M'$  is  $N[Q^\varphi/x^\varphi]$ . Consider a redex in  $M'$ . Either there is corresponding redex in  $N$  with equal cut rank, which is less than  $\text{cr}(M)$  by assumption, or the cut rank equals  $\text{len}(\varphi)$ , which by definition is less than  $\text{cr}((\lambda x^\varphi. N)Q)$ .
3. If  $M$  is of the form

$$\text{case}(\text{in}_i(N^{\varphi_i}), x_1^{\varphi_1}.N_1^X, x_2^{\varphi_2}.N_2^X),$$

then  $M' \equiv N_i[N/x_i^{\varphi_i}]$ . Consider a redex in  $M'$ . Either there is corresponding redex in  $N_i$  with equal cut rank, which is less than  $\text{cr}(M)$  by assumption; or the cut rank equals  $\text{len}(\varphi_i)$ , which by definition is less than  $\text{cr}(\text{case}(\text{in}_i(N^{\varphi_i}), x_1^{\varphi_1}.N_1^X, x_2^{\varphi_2}.N_2^X))$ .  $\square$

**Theorem pty.3.** *All proof terms reduce to normal form; all **derivations** reduce to normal **derivations**.*

*Proof.* The second follows from the first. We prove the first by complete induction on  $m = \text{mr}(M)$ , where  $M$  is a proof term.

1. If  $m = 0$ ,  $M$  is already normal.
2. Otherwise, we proceed by induction on  $n$ , the number of redexes in  $M$  with cut rank equal to  $m$ .

- a) If  $n = 1$ , select any redex  $N$  such that  $m = \text{cr}(N) > \text{cr}(P)$  for any proper sub-term  $P$  which is also a redex of course. Such a redex must exist, since any term only has finitely many subterms.

Let  $N'$  denote the reductum of  $N$ . Now by the lemma  $\text{mr}(N') < \text{mr}(N)$ , thus we can see that  $n$ , the number of redexes with  $\text{cr}(=)m$  is decreased. So  $m$  is decreased (by 1 or more), and we can apply the inductive hypothesis for  $m$ .

- b) For the induction step, assume  $n > 1$ . the process is similar, except that  $n$  is only decreased to a positive number and thus  $m$  does not change. We simply apply the induction hypothesis for  $n$ .  $\square$

The normalization of terms is actually not specific to the reduction order we chose. In fact, one can prove that regardless of the order in which redexes are reduced, the term always reduces to a normal form. This property is called *strong normalization*.

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## Bibliography