

## int.1 Natural Deduction

int:int:ntd:  
sec Natural deduction without the  $\perp_C$  rules is a standard **derivation** system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction **derivations** have undischarged assumptions, we should consider such a **derivation**, say, of  $\varphi$  from **undischarged** assumptions  $\Gamma$ , as a function that turns constructions of all  $\psi \in \Gamma$  into a construction of  $\varphi$ . If there is a **derivation** of  $\varphi$  from no **undischarged** assumptions, then there is a construction of  $\varphi$  in the sense of the BHK interpretation. For the purpose of the discussion, however, we'll suppress the  $\Gamma$  when not needed.

An assumption  $\varphi$  by itself is a **derivation** of  $\varphi$  from the **undischarged** assumption  $\varphi$ . This agrees with the BHK-interpretation: the identity function on constructions turns any construction of  $\varphi$  into a construction of  $\varphi$ .

### Conjunction

$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\text{Intro}$	$\frac{\varphi \wedge \psi}{\varphi} \wedge\text{Elim}$ $\frac{\varphi \wedge \psi}{\psi} \wedge\text{Elim}$
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Suppose we have constructions  $N_1, N_2$  of  $\varphi_1$  and  $\varphi_2$ , respectively. Then we also have a construction  $\varphi_1 \wedge \varphi_2$ , namely the pair  $\langle N_1, N_2 \rangle$ .

A construction of  $\varphi_1 \wedge \varphi_1$  on the BHK interpretation is a pair  $\langle N_1, N_2 \rangle$ . So assume we have such a pair. Then we also have a construction of each conjunct:  $N_1$  is a construction of  $\varphi_1$  and  $N_2$  is a construction of  $\varphi_2$ .

### Conditional

$\frac{[\varphi]^u \quad \dots \quad \psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro}$	$\frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow\text{Elim}$
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If we have a **derivation** of  $\psi$  from **undischarged** assumption  $\varphi$ , then there is a function  $f$  that turns constructions of  $\varphi$  into constructions of  $\psi$ . That same function is a construction of  $\varphi \rightarrow \psi$ . So, if the premise of  $\rightarrow$ Intro has a construction conditional on a construction of  $\varphi$ , the conclusion  $\varphi \rightarrow \psi$  has a construction.

On the other hand, suppose there are constructions  $N$  of  $\varphi$  and  $f$  of  $\varphi \rightarrow \psi$ . A construction of  $\varphi \rightarrow \psi$  is a function that turns constructions of  $\varphi$  into constructions of  $\psi$ . So,  $f(N)$  is a construction of  $\psi$ , i.e., the conclusion of  $\rightarrow$ Elim has a construction.

## Disjunction

$$\begin{array}{c}
 \frac{\varphi}{\varphi \vee \psi} \vee\text{Intro} \\
 \\
 \frac{\psi}{\varphi \vee \psi} \vee\text{Intro}
 \end{array}
 \qquad
 \begin{array}{c}
 [\varphi]^n \qquad [\psi]^n \\
 \vdots \qquad \qquad \vdots \\
 \vdots \qquad \qquad \vdots \\
 \vdots \qquad \qquad \vdots \\
 \vdots \qquad \qquad \vdots \\
 n \frac{\varphi \vee \psi}{\chi} \qquad \frac{\chi}{\chi} \vee\text{Elim}
 \end{array}$$

If we have a construction  $N_i$  of  $\varphi_i$  we can turn it into a construction  $\langle i, N_i \rangle$  of  $\varphi_1 \vee \varphi_2$ . On the other hand, suppose we have a construction of  $\varphi_1 \vee \varphi_2$ , i.e., a pair  $\langle i, N_i \rangle$  where  $N_i$  is a construction of  $\varphi_i$ , and also functions  $f_1, f_2$ , which turn constructions of  $\varphi_1, \varphi_2$ , respectively, into constructions of  $\chi$ . Then  $f_i(N_i)$  is a construction of  $\chi$ , the conclusion of  $\vee$ Elim.

## Absurdity

$$\frac{\perp}{\varphi} \perp_I$$

If we have a **derivation** of  $\perp$  from **undischarged** assumptions  $\psi_1, \dots, \psi_n$ , then there is a function  $f(M_1, \dots, M_n)$  that turns constructions of  $\psi_1, \dots, \psi_n$  into a construction of  $\perp$ . Since  $\perp$  has no construction, there cannot be any constructions of all of  $\psi_1, \dots, \psi_n$  either. Hence,  $f$  also has the property that *if*  $M_1, \dots, M_n$  are constructions of  $\psi_1, \dots, \psi_n$ , respectively, *then*  $f(M_1, \dots, M_n)$  is a construction of  $\varphi$ .

## Rules for $\neg$

Since  $\neg\varphi$  is defined as  $\varphi \rightarrow \perp$ , we strictly speaking do not need rules for  $\neg$ . But if we did, this is what they'd look like:

$$\begin{array}{c}
 [\varphi]^n \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 n \frac{\perp}{\neg\varphi} \neg\text{Intro}
 \end{array}
 \qquad
 \frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}$$

## Examples of Derivations

1.  $\vdash \varphi \rightarrow (\neg\varphi \rightarrow \perp)$ , i.e.,  $\vdash \varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)$

$$\frac{\frac{\frac{[\varphi]^2 \quad \frac{[\varphi \rightarrow \perp]^1}{\perp} \rightarrow\text{Elim}}{(\varphi \rightarrow \perp) \rightarrow \perp} \rightarrow\text{Intro}}{\varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)} \rightarrow\text{Intro}}{\varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)} \rightarrow\text{Intro}$$

2.  $\vdash ((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

$$\frac{\frac{\frac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi} \wedge\text{Intro}}{((\varphi \wedge \psi) \rightarrow \chi)^3} \rightarrow\text{Elim}}{\frac{\frac{\frac{1 \quad \chi}{\psi \rightarrow \chi} \rightarrow\text{Intro}}{\varphi \rightarrow (\psi \rightarrow \chi)} \rightarrow\text{Intro}}{((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))} \rightarrow\text{Intro}}{((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))} \rightarrow\text{Intro}$$

3.  $\vdash \neg(\varphi \wedge \neg\varphi)$ , i.e.,  $\vdash (\varphi \wedge (\varphi \rightarrow \perp)) \rightarrow \perp$

$$\frac{\frac{\frac{[\varphi \wedge (\varphi \rightarrow \perp)]^1}{\varphi \rightarrow \perp} \wedge\text{Elim} \quad \frac{[\varphi \wedge (\varphi \rightarrow \perp)]^1}{\varphi} \wedge\text{Elim}}{\frac{\perp}{(\varphi \wedge (\varphi \rightarrow \perp)) \rightarrow \perp} \rightarrow\text{Intro}}{(\varphi \wedge (\varphi \rightarrow \perp)) \rightarrow \perp} \rightarrow\text{Intro}$$

4.  $\vdash \neg\neg(\varphi \vee \neg\varphi)$ , i.e.,  $\vdash ((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp$

$$\frac{\frac{\frac{[\varphi]^1}{\varphi \vee (\varphi \rightarrow \perp)} \vee\text{Intro}}{((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp)^2} \rightarrow\text{Elim}}{\frac{\frac{\frac{1 \quad \perp}{\varphi \rightarrow \perp} \rightarrow\text{Intro}}{\varphi \vee (\varphi \rightarrow \perp)} \vee\text{Intro}}{((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp} \rightarrow\text{Intro}}{((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp} \rightarrow\text{Intro}$$

**Proposition int.1.** *If  $\Gamma \vdash \varphi$  in intuitionistic logic,  $\Gamma \vdash \varphi$  in classical logic. In particular, if  $\varphi$  is an intuitionistic theorem, it is also a classical theorem.*

*Proof.* Every natural deduction rule is also a rule in classical natural deduction, so every **derivation** in intuitionistic logic is also a **derivation** in classical logic.  $\square$

**Problem int.1.** Give **derivations** in intuitionistic logic of the following **formulas**:

1.  $(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$

2.  $\neg\neg\neg\varphi \rightarrow \neg\varphi$
3.  $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$
4.  $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
5.  $(\neg\varphi \vee \neg\psi) \rightarrow \neg(\varphi \wedge \psi)$
6.  $\neg\neg(\varphi \wedge \psi) \rightarrow (\neg\neg\varphi \vee \neg\neg\psi)$

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## Bibliography