int.1 Natural Deduction

int:int:ntd: sec

Natural deduction without the $\bot_C$ rules is a standard derivation system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction derivations have undischarged assumptions, we should consider such a derivation, say, of φ from undischarged assumptions Γ, as a function that turns constructions of all $\psi \in \Gamma$ into a construction of φ. If there is a derivation of φ from no undischarged assumptions, then there is a construction of φ in the sense of the BHK interpretation. For the purpose of the discussion, however, we'll suppress the Γ when not needed.

An assumption φ by itself is a derivation of φ from the undischarged assumption φ. This agrees with the BHK-interpretation: the identity function on constructions turns any construction of φ into a construction of φ.

Conjunction

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \wedge\text{Intro} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\ \wedge\text{Elim} \qquad \frac{\varphi \wedge \psi}{\psi}\ \wedge\text{Elim}$$

Suppose we have constructions N_1, N_2 of φ_1 and φ_2, respectively. Then we also have a construction $\varphi_1 \wedge \varphi_2$, namely the pair $\langle N_1, N_2 \rangle$.

A construction of $\varphi_1 \wedge \varphi_1$ on the BHK interpretation is a pair $\langle N_1, N_2 \rangle$. So assume we have such a pair. Then we also have a construction of each conjunct: N_1 is a construction of φ_1 and N_2 is a construction of φ_2.

Conditional

$$u\ \frac{\begin{matrix}[\varphi]^u \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\ \to\text{Intro} \qquad\qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\ \to\text{Elim}$$

If we have a derivation of ψ from undischarged assumption φ, then there is a function f that turns constructions of φ into constructions of ψ. That same function is a construction of $\varphi \to \psi$. So, if the premise of $\to$Intro has a construction conditional on a construction of φ, the conclusion $\varphi \to \psi$ has a construction.

On the other hand, suppose there are constructions N of φ and f of $\varphi \to \psi$. A construction of $\varphi \to \psi$ is a function that turns constructions of φ into constructions of ψ. So, $f(N)$ is a construction of ψ, i.e., the conclusion of $\to$Elim has a construction.

Disjunction

$$\frac{\varphi}{\varphi \lor \psi}\ {\lor}\mathrm{Intro} \qquad \frac{\psi}{\varphi \lor \psi}\ {\lor}\mathrm{Intro}$$

$$n\ \frac{\varphi \lor \psi \qquad \begin{matrix}[\varphi]^n \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]^n \\ \vdots \\ \chi\end{matrix}}{\chi}\ {\lor}\mathrm{Elim}$$

If we have a construction N_i of φ_i we can turn it into a construction $\langle i, N_i \rangle$ of $\varphi_1 \lor \varphi_2$. On the other hand, suppose we have a construction of $\varphi_1 \lor \varphi_2$, i.e., a pair $\langle i, N_i \rangle$ where N_i is a construction of φ_i, and also functions f_1, f_2, which turn constructions of φ_1, φ_2, respectively, into constructions of χ. Then $f_i(N_i)$ is a construction of χ, the conclusion of $\lor$Elim.

Absurdity

$$\frac{\bot}{\varphi}\ \bot_I$$

If we have a derivation of $\bot$ from undischarged assumptions $\psi_1, \ldots, \psi_n$, then there is a function $f(M_1, \ldots, M_n)$ that turns constructions of $\psi_1, \ldots, \psi_n$ into a construction of $\bot$. Since $\bot$ has no construction, there cannot be any constructions of all of $\psi_1, \ldots, \psi_n$ either. Hence, f also has the property that *if* $M_1, \ldots, M_n$ are constructions of $\psi_1, \ldots, \psi_n$, respectively, *then* $f(M_1, \ldots, M_n)$ is a construction of φ.

Rules for $\neg$

Since $\neg\varphi$ is defined as $\varphi \to \bot$, we strictly speaking do not need rules for $\neg$. But if we did, this is what they'd look like:

$$n\ \frac{\begin{matrix}[\varphi]^n \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\ \neg\mathrm{Intro} \qquad \frac{\neg\varphi \qquad \varphi}{\bot}\ \neg\mathrm{Elim}$$

Examples of Derivations

1. $\vdash \varphi \to (\neg\varphi \to \bot)$, i.e., $\vdash \varphi \to ((\varphi \to \bot) \to \bot)$

$$
\begin{prooftree}
\AxiomC{$[\varphi]^2$}
\AxiomC{$[\varphi \to \bot]^1$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\varphi \to \bot) \to \bot$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to (\varphi \to \bot) \to \bot$}
\end{prooftree}
$$

2. $\vdash ((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

$$
\begin{prooftree}
\AxiomC{$[(\varphi \wedge \psi) \to \chi]^3$}
\AxiomC{$[\varphi]^2$}
\AxiomC{$[\psi]^1$}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$\varphi \wedge \psi$}
\RightLabel{$\to$Elim}
\BinaryInfC{χ}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$\psi \to \chi$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to (\psi \to \chi)$}
\LeftLabel{3}
\RightLabel{$\to$Intro}
\UnaryInfC{$((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$}
\end{prooftree}
$$

3. $\vdash \neg(\varphi \wedge \neg\varphi)$, i.e., $\vdash (\varphi \wedge (\varphi \to \bot)) \to \bot$

$$
\begin{prooftree}
\AxiomC{$[\varphi \wedge (\varphi \to \bot)]^1$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{$\varphi \to \bot$}
\AxiomC{$[\varphi \wedge (\varphi \to \bot)]^1$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{φ}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\varphi \wedge (\varphi \to \bot)) \to \bot$}
\end{prooftree}
$$

4. $\vdash \neg\neg(\varphi \vee \neg\varphi)$, i.e., $\vdash ((\varphi \vee (\varphi \to \bot)) \to \bot) \to \bot$

$$
\begin{prooftree}
\AxiomC{$[(\varphi \vee (\varphi \to \bot)) \to \bot]^2$}
\AxiomC{$[(\varphi \vee (\varphi \to \bot)) \to \bot]^2$}
\AxiomC{$[\varphi]^1$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$\varphi \vee (\varphi \to \bot)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to \bot$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$\varphi \vee (\varphi \to \bot)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$((\varphi \vee (\varphi \to \bot)) \to \bot) \to \bot$}
\end{prooftree}
$$

Proposition int.1. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every natural deduction rule is also a rule in classical natural deduction, so every derivation in intuitionistic logic is also a derivation in classical logic. □

Problem int.1. Give derivations in intutionistic logic of the following.

1. $(\neg\varphi \vee \psi) \to (\varphi \to \psi)$
2. $\neg\neg\neg\varphi \to \neg\varphi$

3. $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$

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Bibliography