

## int.1 The Brouwer-Heyting-Kolmogorov Interpretation

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sec

Proofs of validity of intuitionistic propositions using the BHK interpretation are confusing; they have to be explained better.

There is an informal constructive interpretation of the intuitionist connectives, usually known as the Brouwer-Heyting-Kolmogorov interpretation. It uses the notion of a “construction,” which you may think of as a constructive proof. (We don’t use “proof” in the BHK interpretation so as not to get confused with the notion of a [derivation](#) in a formal proof system.) Based on this intuitive notion, the BHK interpretation explains the meanings of the intuitionistic connectives.

1. We assume that we know what constitutes a construction of an atomic statement.
2. A construction of  $\varphi_1 \wedge \varphi_2$  is a pair  $\langle M_1, M_2 \rangle$  where  $M_1$  is a construction of  $\varphi_1$  and  $M_2$  is a construction of  $\varphi_2$ .
3. A construction of  $\varphi_1 \vee \varphi_2$  is a pair  $\langle s, M \rangle$  where  $s$  is 1 and  $M$  is a construction of  $\varphi_1$ , or  $s$  is 2 and  $M$  is a construction of  $\varphi_2$ .
4. A construction of  $\varphi \rightarrow \psi$  is a function that converts a construction of  $\varphi$  into a construction of  $\psi$ .
5. There is no construction for  $\perp$  (absurdity).
6.  $\neg\varphi$  is defined as synonym for  $\varphi \rightarrow \perp$ . That is, a construction of  $\neg\varphi$  is a function converting a construction of  $\varphi$  into a construction of  $\perp$ .

**Example int.1.** Take  $\neg\perp$  for example. A construction of it is a function which, given any construction of  $\perp$  as input, provides a construction of  $\perp$  as output. Obviously, the identity function  $\text{Id}$  is such a construction: given a construction  $M$  of  $\perp$ ,  $\text{Id}(M) = M$  yields a construction of  $\perp$ .

Generally speaking,  $\neg\varphi$  means “A construction of  $\varphi$  is impossible”.

**Example int.2.** Let us prove  $\varphi \rightarrow \neg\neg\varphi$  for any proposition  $\varphi$ , which is  $\varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)$ . The construction should be a function  $f$  that, given a construction  $M$  of  $\varphi$ , returns a construction  $f(M)$  of  $(\varphi \rightarrow \perp) \rightarrow \perp$ . Here is how  $f$  constructs the construction of  $(\varphi \rightarrow \perp) \rightarrow \perp$ : We have to define a function  $g$  which, when given a construction  $h$  of  $\varphi \rightarrow \perp$  as input, outputs a construction of  $\perp$ . We can define  $g$  as follows: apply the input  $h$  to the construction  $M$  of  $\varphi$  (that we received earlier). Since the output  $h(M)$  of  $h$  is a construction of  $\perp$ ,  $f(M)(h) = h(M)$  is a construction of  $\perp$  if  $M$  is a construction of  $\varphi$ .

**Example int.3.** Let us give a construction for  $\neg(\varphi \wedge \neg\varphi)$ , i.e.,  $(\varphi \wedge (\varphi \rightarrow \perp)) \rightarrow \perp$ . This is a function  $f$  which, given as input a construction  $M$  of  $\varphi \wedge (\varphi \rightarrow \perp)$ , yields a construction of  $\perp$ . A construction of a conjunction  $\psi_1 \wedge \psi_2$  is a pair  $\langle N_1, N_2 \rangle$  where  $N_1$  is a construction of  $\psi_1$  and  $N_2$  is a construction of  $\psi_2$ . We can define functions  $p_1$  and  $p_2$  which recover from a construction of  $\psi_1 \wedge \psi_2$  the constructions of  $\psi_1$  and  $\psi_2$ , respectively:

$$\begin{aligned} p_1(\langle N_1, N_2 \rangle) &= N_1 \\ p_2(\langle N_1, N_2 \rangle) &= N_2 \end{aligned}$$

Here is what  $f$  does: First it applies  $p_1$  to its input  $M$ . That yields a construction of  $\varphi$ . Then it applies  $p_2$  to  $M$ , yielding a construction of  $\varphi \rightarrow \perp$ . Such a construction, in turn, is a function  $p_2(M)$  which, if given as input a construction of  $\varphi$ , yields a construction of  $\perp$ . In other words, if we apply  $p_2(M)$  to  $p_1(M)$ , we get a construction of  $\perp$ . Thus, we can define  $f(M) = p_2(M)(p_1(M))$ .

**Example int.4.** Let us give a construction of  $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$ , i.e., a function  $f$  which turns a construction  $g$  of  $(\varphi \wedge \psi) \rightarrow \chi$  into a construction of  $(\varphi \rightarrow (\psi \rightarrow \chi))$ . The construction  $g$  is itself a function (from constructions of  $\varphi \wedge \psi$  to constructions of  $\chi$ ). And the output  $f(g)$  is a function  $h_g$  from constructions of  $\varphi$  to functions from constructions of  $\psi$  to constructions of  $\chi$ .

Ok, this is confusing. We have to construct a certain function  $h_g$ , which will be the output of  $f$  for input  $g$ . The input of  $h_g$  is a construction  $M$  of  $\varphi$ . The output of  $h_g(M)$  should be a function  $k_M$  from constructions  $N$  of  $\psi$  to constructions of  $\chi$ . Let  $k_{g,M}(N) = g(\langle M, N \rangle)$ . Remember that  $\langle M, N \rangle$  is a construction of  $\varphi \wedge \psi$ . So  $k_{g,M}$  is a construction of  $\psi \rightarrow \chi$ : it maps constructions  $N$  of  $\psi$  to constructions of  $\chi$ . Now let  $h_g(M) = k_{g,M}$ . That's a function that maps constructions  $M$  of  $\varphi$  to constructions  $k_{g,M}$  of  $\psi \rightarrow \chi$ . Now let  $f(g) = h_g$ . That's a function that maps constructions  $g$  of  $(\varphi \wedge \psi) \rightarrow \chi$  to constructions of  $\varphi \rightarrow (\psi \rightarrow \chi)$ . Whew!

The statement  $\varphi \vee \neg\varphi$  is called the Law of Excluded Middle. We can prove it for some specific  $\varphi$  (e.g.,  $\perp \vee \neg\perp$ ), but not in general. This is because the intuitionistic disjunction requires a construction of one of the disjuncts, but there are statements which currently can neither be proved nor refuted (say, Goldbach's conjecture). However, you can't refute the law of excluded middle either: that is,  $\neg\neg(\varphi \vee \neg\varphi)$  holds.

**Example int.5.** To prove  $\neg\neg(\varphi \vee \neg\varphi)$ , we need a function  $f$  that transforms a construction of  $\neg(\varphi \vee \neg\varphi)$ , i.e., of  $(\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp$ , into a construction of  $\perp$ . In other words, we need a function  $f$  such that  $f(g)$  is a construction of  $\perp$  if  $g$  is a construction of  $\neg(\varphi \vee \neg\varphi)$ .

Suppose  $g$  is a construction of  $\neg(\varphi \vee \neg\varphi)$ , i.e., a function that transforms a construction of  $\varphi \vee \neg\varphi$  into a construction of  $\perp$ . A construction of  $\varphi \vee \neg\varphi$  is a pair  $\langle s, M \rangle$  where either  $s = 1$  and  $M$  is a construction of  $\varphi$ , or  $s = 2$  and  $M$  is a construction of  $\neg\varphi$ . Let  $h_1$  be the function mapping a construction  $M_1$  of  $\varphi$  to a construction of  $\varphi \vee \neg\varphi$ : it maps  $M_1$  to  $\langle 1, M_1 \rangle$ . And let  $h_2$  be the function

mapping a construction  $M_2$  of  $\neg\varphi$  to a construction of  $\varphi \vee \neg\varphi$ : it maps  $M_2$  to  $\langle 2, M_2 \rangle$ .

Let  $k$  be  $g \circ h_1$ : it is a function which, if given a construction of  $\varphi$ , returns a construction of  $\perp$ , i.e., it is a construction of  $\varphi \rightarrow \perp$  or  $\neg\varphi$ . Now let  $l$  be  $g \circ h_2$ . It is a function which, given a construction of  $\neg\varphi$ , provides a construction of  $\perp$ . Since  $k$  is a construction of  $\neg\varphi$ ,  $l(k)$  is a construction of  $\perp$ .

Together, what we've done is describe how we can turn a construction  $g$  of  $\neg(\varphi \vee \neg\varphi)$  into a construction of  $\perp$ , i.e., the function  $f$  mapping a construction  $g$  of  $\neg(\varphi \vee \neg\varphi)$  to the construction  $l(k)$  of  $\perp$  is a construction of  $\neg\neg(\varphi \vee \neg\varphi)$ .

As you can see, using the BHK interpretation to show the intuitionistic validity of **formulas** quickly becomes cumbersome and confusing. Luckily, there are better **derivation** systems for intuitionistic logic, and more precise semantic interpretations.

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## Bibliography