

tcp.1 Theories in which \mathbf{Q} is Interpretable are Undecidable

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sec We can strengthen these results even more. Informally, an interpretation of a language \mathcal{L}_1 in another language \mathcal{L}_2 involves defining the universe, relation symbols, and function symbols of \mathcal{L}_1 with **formulas** in \mathcal{L}_2 . Though we won't take the time to do this, one can make this definition precise.

Theorem tcp.1. *Suppose \mathbf{T} is a theory in a language in which one can interpret the language of arithmetic, in such a way that \mathbf{T} is consistent with the interpretation of \mathbf{Q} . Then \mathbf{T} is undecidable. If \mathbf{T} proves the interpretation of the axioms of \mathbf{Q} , then no consistent extension of \mathbf{T} is decidable.*

The proof is just a small modification of the proof of the last theorem; one could use a counterexample to get a separation of \mathbf{Q} and $\bar{\mathbf{Q}}$. One can take **ZFC**, Zermelo–Fraenkel set theory with the axiom of choice, to be an axiomatic foundation that is powerful enough to carry out a good deal of ordinary mathematics. In **ZFC** one can define the natural numbers, and via this interpretation, the axioms of \mathbf{Q} are true. So we have

Corollary tcp.2. *There is no decidable extension of **ZFC**.*

Corollary tcp.3. *There is no complete, consistent, computably **axiomatizable** extension of **ZFC**.*

The language of **ZFC** has only a single binary relation, \in . (In fact, you don't even need equality.) So we have

Corollary tcp.4. *First-order logic for any language with a binary relation symbol is undecidable.*

This result extends to any language with two unary function symbols, since one can use these to simulate a binary relation symbol. The results just cited are tight: it turns out that first-order logic for a language with only *unary* relation symbols and at most one *unary* function symbol is decidable.

One more bit of trivia. We know that the set of sentences in the language $0, ', +, \times, <$ true in the standard model is undecidable. In fact, one can define $<$ in terms of the other symbols, and then one can define $+$ in terms of \times and $'$. So the set of true sentences in the language $0, ', \times$ is undecidable. On the other hand, Presburger has shown that the set of sentences in the language $0, ', +$ true in the language of arithmetic is decidable. The procedure is computationally infeasible, however.

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Bibliography