tcp.1 Q has no Complete, Consistent, Axiomatizable Extensions

inc:tcp:inc: sec

inc:tcp:inc: thm:first-incompleteness **Theorem tcp.1.** *There is no complete, consistent, axiomatizable extension of* **Q**.

Proof. We already know that there is no consistent, decidable extension of **Q**. But if **T** is complete and axiomatized, then it is decidable. □

This theorems is not that far from Gödel's original 1931 formulation of the First Incompleteness Theorem. Aside from the more modern terminology, the key differences are this: Gödel has "ω-consistent" instead of "consistent"; and he could not say "axiomatizable" in full generality, since the formal notion of computability was not in place yet. (The formal models of computability were developed over the following decade, including by Gödel, and in large part to be able to characterize the kinds of theories that are susceptible to the Gödel phenomenon.) explanation

The theorem says you can't have it all, namely, completeness, consistency, and axiomatizability. If you give up any one of these, though, you can have the other two: **Q** is consistent and computably axiomatized, but not complete; the inconsistent theory is complete, and computably axiomatized (say, by $\{0 \neq 0\}$), but not consistent; and the set of true sentence of arithmetic is complete and consistent, but it is not computably axiomatized.

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Bibliography