The following theorem says that not only is $Q$ undecidable, but, in fact, any theory that does not disagree with $Q$ is undecidable.

**Theorem tcp.1.** Let $T$ be any theory in the language of arithmetic that is consistent with $Q$ (i.e., $T \cup Q$ is consistent). Then $T$ is undecidable.

**Proof.** Remember that $Q$ has a finite set of axioms, $Q_1, \ldots, Q_8$. We can even replace these by a single axiom, $\alpha = Q_1 \land \cdots \land Q_8$.

Suppose $T$ is a decidable theory consistent with $Q$. Let

$$C = \{ \varphi : T \vdash \alpha \rightarrow \varphi \}.$$  

We show that $C$ would be a computable separation of $Q$ and $\overline{Q}$, a contradiction. First, if $\varphi$ is in $Q$, then $\varphi$ is provable from the axioms of $Q$; by the deduction theorem, there is a derivation of $\alpha \rightarrow \varphi$ in first-order logic. So $\varphi$ is in $C$.

On the other hand, if $\varphi$ is in $\overline{Q}$, then there is a proof of $\alpha \rightarrow \neg \varphi$ in first-order logic. If $T$ also proves $\alpha \rightarrow \varphi$, then $T$ proves $\neg \alpha$, in which case $T \cup Q$ is inconsistent. But we are assuming $T \cup Q$ is consistent, so $T$ does not prove $\alpha \rightarrow \varphi$, and so $\varphi$ is not in $C$.

We’ve shown that if $\varphi$ is in $Q$, then it is in $C$, and if $\varphi$ is in $\overline{Q}$, then it is in $\overline{C}$. So $C$ is a computable separation, which is the contradiction we were looking for.

This theorem is very powerful. For example, it implies:

**Corollary tcp.2.** First-order logic for the language of arithmetic (that is, the set $\{ \varphi : \varphi$ is provable in first-order logic$\}$) is undecidable.

**Proof.** First-order logic is the set of consequences of $\emptyset$, which is consistent with $Q$. \hfill \Box

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**Bibliography**