

tcp.1 Theories Consistent with \mathbf{Q} are Undecidable

inc:tcp:con:
sec The following theorem says that not only is \mathbf{Q} undecidable, but, in fact, any theory that does not disagree with \mathbf{Q} is undecidable.

Theorem tcp.1. *Let \mathbf{T} be any theory in the language of arithmetic that is consistent with \mathbf{Q} (i.e., $\mathbf{T} \cup \mathbf{Q}$ is consistent). Then \mathbf{T} is undecidable.*

Proof. Remember that \mathbf{Q} has a finite set of axioms, Q_1, \dots, Q_8 . We can even replace these by a single axiom, $\alpha = Q_1 \wedge \dots \wedge Q_8$.

Suppose \mathbf{T} is a decidable theory consistent with \mathbf{Q} . Let

$$C = \{\varphi : \mathbf{T} \vdash \alpha \rightarrow \varphi\}.$$

We show that C would be a computable separation of \mathbf{Q} and $\bar{\mathbf{Q}}$, a contradiction. First, if φ is in \mathbf{Q} , then φ is provable from the axioms of \mathbf{Q} ; by the deduction theorem, there is a **derivation** of $\alpha \rightarrow \varphi$ in first-order logic. So φ is in C .

On the other hand, if φ is in $\bar{\mathbf{Q}}$, then there is a proof of $\alpha \rightarrow \neg\varphi$ in first-order logic. If \mathbf{T} also proves $\alpha \rightarrow \varphi$, then \mathbf{T} proves $\neg\alpha$, in which case $\mathbf{T} \cup \mathbf{Q}$ is inconsistent. But we are assuming $\mathbf{T} \cup \mathbf{Q}$ is consistent, so \mathbf{T} does not prove $\alpha \rightarrow \varphi$, and so φ is not in C .

We've shown that if φ is in \mathbf{Q} , then it is in C , and if φ is in $\bar{\mathbf{Q}}$, then it is in \bar{C} . So C is a computable separation, which is the contradiction we were looking for. \square

This theorem is very powerful. For example, it implies:

Corollary tcp.2. *First-order logic for the language of arithmetic (that is, the set $\{\varphi : \varphi \text{ is provable in first-order logic}\}$) is undecidable.*

Proof. First-order logic is the set of consequences of \emptyset , which is consistent with \mathbf{Q} . \square

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Bibliography