tcp.1 Axiomatizable Theories

inc:tcp:cax:sec A theory $\mathbf{T}$ is said to be *axiomatizable* if it has a computable set of axioms A. (Saying that A is a set of axioms for $\mathbf{T}$ means $T = \{\varphi : A \vdash \varphi\}$.) Any "reasonable" axiomatization of the natural numbers will have this property. In particular, any theory with a finite set of axioms is axiomatizable.

Lemma tcp.1. *Suppose $\mathbf{T}$ is axiomatizable. Then $\mathbf{T}$ is computably enumerable.*

Proof. Suppose A is a computable set of axioms for $\mathbf{T}$. To determine if $\varphi \in T$, just search for a proof of φ from the axioms.

Put slightly differently, φ is in $\mathbf{T}$ if and only if there is a finite list of axioms $\psi_1, \dots, \psi_k$ in A and a proof of $(\psi_1 \wedge \dots \wedge \psi_k) \to \varphi$ in first-order logic. But we already know that any set with a definition of the form "there exists ... such that ..." is c.e., provided the second "..." is computable. $\square$

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Bibliography