req.1 Undecidability

We call a theory $T$ undecidable if there is no computational procedure which, after finitely many steps and unfailingly, provides a correct answer to the question “does $T$ prove $\varphi$?” for any sentence $\varphi$ in the language of $T$. So $Q$ would be decidable iff there were a computational procedure which decides, given a sentence $\varphi$ in the language of arithmetic, whether $Q \vdash \varphi$ or not. We can make this more precise by asking: Is the relation $\text{Prov}_Q(y)$, which holds of $y$ iff $y$ is the Gödel number of a sentence provable in $Q$, recursive? The answer is: no.

**Theorem req.1.** $Q$ is undecidable, i.e., the relation

$$\text{Prov}_Q(y) \iff \exists x \text{ Prf}_Q(x, y)$$

is not recursive.

**Proof.** Suppose it were. Then we could solve the halting problem as follows: Given $e$ and $n$, we know that $\varphi_e(n) \downarrow$ iff there is an $s$ such that $T(e, n, s)$, where $T$ is Kleene’s predicate from ??, since $T$ is primitive recursive it is representable in $Q$ by a formula $\psi_T$, that is, $Q \vdash \psi_T(\overline{e}, \overline{n}, \overline{s})$ iff $T(e, n, s)$. If $Q \vdash \psi_T(\overline{e}, \overline{n}, \overline{s})$ then also $Q \vdash \exists y \psi_T(\overline{e}, \overline{n}, y)$. If no such $s$ exists, then $Q \vdash \neg \psi_T(\overline{e}, \overline{n}, \overline{s})$ for every $s$. But $Q$ is $\omega$-consistent, i.e., if $Q \vdash \neg \varphi(n)$ for every $n \in \mathbb{N}$, then $Q \not\vdash \exists y \varphi(y)$. We know this because the axioms of $Q$ are true in the standard model $\mathbb{N}$. So, $Q \not\vdash \exists y \psi_T(\overline{e}, \overline{n}, y)$. In other words, $Q \vdash \exists y \psi_T(\overline{e}, \overline{n}, y)$ iff there is an $s$ such that $T(e, n, s)$, i.e., iff $\varphi_e(n) \downarrow$. From $e$ and $n$ we can compute $\exists y \psi_T(\overline{e}, \overline{n}, y)^\#$, let $g(e, n)$ be the primitive recursive function which does that. So

$$h(e, n) = \begin{cases} 1 & \text{ if } \text{Prov}_Q(g(e, n)) \\ 0 & \text{ otherwise.} \end{cases}$$

This would show that $h$ is recursive if $\text{Prov}_Q$ is. But $h$ is not recursive, by ??, so $\text{Prov}_Q$ cannot be either. \[\square\]

**Corollary req.2.** First-order logic is undecidable.

**Proof.** If first-order logic were decidable, provability in $Q$ would be as well, since $Q \vdash \varphi$ iff $\vdash \varphi$, where $\omega$ is the conjunction of the axioms of $Q$. \[\square\]

Photo Credits

Bibliography