req.1  Representing Relations

Let us say what it means for a relation to be representable.

Definition req.1. A relation \( R(x_0, \ldots, x_k) \) on the natural numbers is representable in \( Q \) if there is a formula \( \varphi_R(x_0, \ldots, x_k) \) such that whenever \( R(n_0, \ldots, n_k) \) is true, \( Q \) proves \( \varphi_R(n_0, \ldots, n_k) \), and whenever \( R(n_0, \ldots, n_k) \) is false, \( Q \) proves \( \neg \varphi_R(n_0, \ldots, n_k) \).

Theorem req.2. A relation is representable in \( Q \) if and only if it is computable.

Proof. For the forwards direction, suppose \( R(x_0, \ldots, x_k) \) is represented by the formula \( \varphi_R(x_0, \ldots, x_k) \). Here is an algorithm for computing \( R \): on input \( n_0, \ldots, n_k \), simultaneously search for a proof of \( \varphi_R(n_0, \ldots, n_k) \) and a proof of \( \neg \varphi_R(n_0, \ldots, n_k) \). By our hypothesis, the search is bound to find one or the other; if it is the first, report “yes,” and otherwise, report “no.”

In the other direction, suppose \( R(x_0, \ldots, x_k) \) is computable. By definition, this means that the function \( \chi_R(x_0, \ldots, x_k) \) is computable. By ??, \( \chi_R \) is represented by a formula, say \( \varphi_{\chi_R}(x_0, \ldots, x_k, y) \). Let \( \varphi_R(x_0, \ldots, x_k) \) be the formula \( \varphi_{\chi_R}(x_0, \ldots, x_k, \top) \). Then for any \( n_0, \ldots, n_k \), if \( R(n_0, \ldots, n_k) \) is true, then \( \chi_R(n_0, \ldots, n_k) = 1 \), in which case \( Q \) proves \( \varphi_{\chi_R}(n_0, \ldots, n_k, \top) \), and so \( Q \) proves \( \varphi_R(n_0, \ldots, n_k) \). On the other hand, if \( R(n_0, \ldots, n_k) \) is false, then \( \chi_R(n_0, \ldots, n_k) = 0 \). This means that \( Q \) proves

\[
\forall y (\varphi_{\chi_R}(n_0, \ldots, n_k, y) \rightarrow y = \top).
\]

Since \( Q \) proves \( \top \neq \top \), \( Q \) proves \( \neg \varphi_{\chi_R}(n_0, \ldots, n_k, \top) \), and so it proves \( \neg \varphi_R(n_0, \ldots, n_k) \).

Problem req.1. Show that if \( R \) is representable in \( Q \), so is \( \chi_R \).

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Bibliography