req.1 Representing Relations

inc:req:rel:sec Let us say what it means for a *relation* to be representable.

inc:req:rel:defn:representing-relations **Definition req.1.** A relation $R(x_0, \dots, x_k)$ on the natural numbers is *representable in* $\mathbf{Q}$ if there is a formula $\varphi_R(x_0, \dots, x_k)$ such that whenever $R(n_0, \dots, n_k)$ is true, $\mathbf{Q}$ proves $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$, and whenever $R(n_0, \dots, n_k)$ is false, $\mathbf{Q}$ proves $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$.

inc:req:rel:thm:representing-rels **Theorem req.2.** *A relation is representable in* $\mathbf{Q}$ *if and only if it is computable.*

Proof. For the forwards direction, suppose $R(x_0, \dots, x_k)$ is represented by the formula $\varphi_R(x_0, \dots, x_k)$. Here is an algorithm for computing R: on input n_0, …, n_k, simultaneously search for a proof of $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ and a proof of $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. By our hypothesis, the search is bound to find one or the other; if it is the first, report "yes," and otherwise, report "no."

In the other direction, suppose $R(x_0, \dots, x_k)$ is computable. By definition, this means that the function $\chi_R(x_0, \dots, x_k)$ is computable. By **??**, χ_R is represented by a formula, say $\varphi_{\chi_R}(x_0, \dots, x_k, y)$. Let $\varphi_R(x_0, \dots, x_k)$ be the formula $\varphi_{\chi_R}(x_0, \dots, x_k, \overline{1})$. Then for any n_0, …, n_k, if $R(n_0, \dots, n_k)$ is true, then $\chi_R(n_0, \dots, n_k) = 1$, in which case $\mathbf{Q}$ proves $\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$, and so $\mathbf{Q}$ proves $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. On the other hand, if $R(n_0, \dots, n_k)$ is false, then $\chi_R(n_0, \dots, n_k) = 0$. This means that $\mathbf{Q}$ proves

$$\forall y\,(\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, y) \to y = \overline{0}).$$

Since $\mathbf{Q}$ proves $\overline{0} \neq \overline{1}$, $\mathbf{Q}$ proves $\lnot\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$, and so it proves $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. □

Problem req.1. Show that if R is representable in $\mathbf{Q}$, so is χ_R.

Photo Credits

Bibliography