## req.1 Representing Relations

inc:req:rel: sec

Let us say what it means for a *relation* to be representable.

inc:req:rel: defn:representing-relations

**Definition req.1.** A relation $R(x_0, \dots, x_k)$ on the natural numbers is *representable in* **Q** if there is a formula $\varphi_R(x_0, \dots, x_k)$ such that whenever $R(n_0, \dots, n_k)$ is true, **Q** proves $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$, and whenever $R(n_0, \dots, n_k)$ is false, **Q** proves $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$.

*inc:req:rel: thm:representing-rels*

**Theorem req.2.** *A relation is representable in* **Q** *if and only if it is computable.*

*Proof.* For the forwards direction, suppose $R(x_0, \dots, x_k)$ is represented by the formula $\varphi_R(x_0, \dots, x_k)$. Here is an algorithm for computing $R$: on input $n_0$, …, $n_k$, simultaneously search for a proof of $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ and a proof of $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. By our hypothesis, the search is bound to find one or the other; if it is the first, report "yes," and otherwise, report "no."

In the other direction, suppose $R(x_0, \dots, x_k)$ is computable. By definition, this means that the function $\chi_R(x_0, \dots, x_k)$ is computable. By **??**, $\chi_R$ is represented by a formula, say $\varphi_{\chi_R}(x_0, \dots, x_k, y)$. Let $\varphi_R(x_0, \dots, x_k)$ be the formula $\varphi_{\chi_R}(x_0, \dots, x_k, \overline{1})$. Then for any $n_0$, …, $n_k$, if $R(n_0, \dots, n_k)$ is true, then $\chi_R(n_0, \dots, n_k) = 1$, in which case **Q** proves $\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$, and so **Q** proves $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. On the other hand, if $R(n_0, \dots, n_k)$ is false, then $\chi_R(n_0, \dots, n_k) = 0$. This means that **Q** proves

$$\forall y\, (\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, y) \to y = \overline{0}).$$

Since **Q** proves $\overline{0} \neq \overline{1}$, **Q** proves $\lnot\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$, and so it proves $\lnot\varphi_R(\overline{n_0}, \dots, \overline{n_k})$. □

**Problem req.1.** Show that if $R$ is representable in **Q**, so is $\chi_R$.

## Photo Credits

## Bibliography