req.1 Regular Minimization is Representable in Q

inc:req:min:sec Let's consider unbounded search. Suppose $g(x, z)$ is regular and representable in **Q**, say by the formula $\varphi_g(x, z, y)$. Let f be defined by $f(z) = \mu x\, [g(x, z) = 0]$. We would like to find a formula $\varphi_f(z, y)$ representing f. The value of $f(z)$ is that number x which (a) satisfies $g(x, z) = 0$ and (b) is the least such, i.e., for any $w < x$, $g(w, z) \neq 0$. So the following is a natural choice:

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \land \forall w\, (w < y \rightarrow \lnot\varphi_g(w, z, 0)).$$

In the general case, of course, we would have to replace z with $z_0, \dots, z_k$.

The proof, again, will involve some lemmas about things **Q** is strong enough to prove.

inc:req:min:lem:succ **Lemma req.1.** *For every constant symbol a and every natural number n,*

$$\mathbf{Q} \vdash (a' + \overline{n}) = (a + \overline{n})'.$$

Proof. The proof is, as usual, by induction on n. In the base case, $n = 0$, we need to show that **Q** proves $(a' + \mathsf{o}) = (a + \mathsf{o})'$. But we have:

inc:req:min:step1 inc:req:min:step2 inc:req:min:step3

$$\begin{aligned}
&\mathbf{Q} \vdash (a' + \mathsf{o}) = a' \quad \text{by axiom } Q_4 && (1)\\
&\mathbf{Q} \vdash (a + \mathsf{o}) = a \quad \text{by axiom } Q_4 && (2)\\
&\mathbf{Q} \vdash (a + \mathsf{o})' = a' \quad \text{by eq. (2)} && (3)\\
&\mathbf{Q} \vdash (a' + \mathsf{o}) = (a + \mathsf{o})' \quad \text{by eq. (1) and eq. (3)}
\end{aligned}$$

In the induction step, we can assume that we have shown that $\mathbf{Q} \vdash (a' + \overline{n}) = (a + \overline{n})'$. Since $\overline{n+1}$ is $\overline{n}'$, we need to show that **Q** proves $(a' + \overline{n}') = (a + \overline{n}')'$. We have:

inc:req:min:step5 inc:req:min:step6

$$\begin{aligned}
&\mathbf{Q} \vdash (a' + \overline{n}') = (a' + \overline{n})' \quad \text{by axiom } Q_5 && (4)\\
&\mathbf{Q} \vdash (a' + \overline{n}') = (a + \overline{n}')' \quad \text{inductive hypothesis} && (5)\\
&\mathbf{Q} \vdash (a' + \overline{n})' = (a + \overline{n}')' \quad \text{by eq. (4) and eq. (5).}
\end{aligned}$$ □

It is again worth mentioning that this is weaker than saying that **Q** proves $\forall x\, \forall y\, (x' + y) = (x + y)'$. Although this sentence is true in $\mathfrak{N}$, **Q** does not prove it.

inc:req:min:lem:less-zero **Lemma req.2.** $\mathbf{Q} \vdash \forall x\, \lnot x < \mathsf{o}$.

Proof. We give the proof informally (i.e., only giving hints as to how to construct the formal derivation).

We have to prove $\lnot a < \mathsf{o}$ for an arbitrary a. By the definition of $<$, we need to prove $\lnot\exists y\, (y' + a) = \mathsf{o}$ in **Q**. We'll assume $\exists y\, (y' + a) = \mathsf{o}$ and prove a contradiction. Suppose $(b' + a) = \mathsf{o}$. Using Q_3, we have that $a = \mathsf{o} \lor \exists y\, a = y'$. We distinguish cases.

Case 1: $a = \mathsf{o}$ holds. From $(b' + a) = \mathsf{o}$, we have $(b' + \mathsf{o}) = \mathsf{o}$. By axiom Q_4 of $\mathbf{Q}$, we have $(b' + \mathsf{o}) = b'$, and hence $b' = \mathsf{o}$. But by axiom Q_2 we also have $b' \neq \mathsf{o}$, a contradiction.

Case 2: For some c, $a = c'$. But then we have $(b' + c') = \mathsf{o}$. By axiom Q_5, we have $(b' + c)' = \mathsf{o}$, again contradicting axiom Q_2. □

Lemma req.3. *For every natural number n,* inc:req:min: lem:less-nsucc

$$\mathbf{Q} \vdash \forall x\,(x < \overline{n+1} \to (x = \mathsf{o} \lor \dots \lor x = \overline{n})).$$

Proof. We use induction on n. Let us consider the base case, when $n = 0$. In that case, we need to show $a < \overline{1} \to a = \mathsf{o}$, for arbitrary a. Suppose $a < \overline{1}$. Then by the defining axiom for $<$, we have $\exists y\,(y' + a) = \mathsf{o}'$ (since $\overline{1} \equiv \mathsf{o}'$).

Suppose b has that property, i.e., we have $(b' + a) = \mathsf{o}'$. We need to show $a = \mathsf{o}$. By axiom Q_3, we have either $a = \mathsf{o}$ or that there is a c such that $a = c'$. In the former case, there is nothing to show. So suppose $a = c'$. Then we have $(b' + c') = \mathsf{o}'$. By axiom Q_5 of $\mathbf{Q}$, we have $(b' + c)' = \mathsf{o}'$. By axiom Q_1, we have $(b' + c) = \mathsf{o}$. But this means, by axiom Q_8, that $c < \mathsf{o}$, contradicting Lemma req.2.

Now for the inductive step. We prove the case for $n + 1$, assuming the case for n. So suppose $a < \overline{n+2}$. Again using Q_3 we can distinguish two cases: $a = \mathsf{o}$ and for some b, $a = c'$. In the first case, $a = \mathsf{o} \lor \dots \lor a = \overline{n+1}$ follows trivially. In the second case, we have $c' < \overline{n+2}$, i.e., $c' < \overline{n+1}'$. By axiom Q_8, for some d, $(d' + c') = \overline{n+1}'$. By axiom Q_5, $(d' + c)' = \overline{n+1}'$. By axiom Q_1, $(d' + c) = \overline{n+1}$, and so $c < \overline{n+1}$ by axiom Q_8. By inductive hypothesis, $c = \mathsf{o} \lor \dots \lor c = \overline{n}$. From this, we get $c' = \mathsf{o}' \lor \dots \lor c' = \overline{n}'$ by logic, and so $a = \overline{1} \lor \dots \lor a = \overline{n+1}$ since $a = c'$. □

Lemma req.4. *For every $m \in \mathbb{N}$,* inc:req:min: lem:trichotomy

$$\mathbf{Q} \vdash \forall y\,((y < \overline{m} \lor \overline{m} < y) \lor y = \overline{m}).$$

Proof. By induction on m. First, consider the case $m = 0$. $\mathbf{Q} \vdash \forall y\,(y = \mathsf{o} \lor \exists z\, y = z')$ by Q_3. Let a be arbitrary. Then either $a = \mathsf{o}$ or for some b, $a = b'$. In the former case, we also have $(a < \mathsf{o} \lor \mathsf{o} < a) \lor a = \mathsf{o}$. But if $a = b'$, then $(b' + \mathsf{o}) = (a + \mathsf{o})$ by the logic of $=$. By Q_4, $(a + \mathsf{o}) = a$, so we have $(b' + \mathsf{o}) = a$, and hence $\exists z\,(z' + \mathsf{o}) = a$. By the definition of $<$ in Q_8, $\mathsf{o} < a$. If $\mathsf{o} < a$, then also $(\mathsf{o} < a \lor a < \mathsf{o}) \lor a = \mathsf{o}$.

Now suppose we have

$$\mathbf{Q} \vdash \forall y\,((y < \overline{m} \lor \overline{m} < y) \lor y = \overline{m})$$

and we want to show

$$\mathbf{Q} \vdash \forall y\,((y < \overline{m+1} \lor \overline{m+1} < y) \lor y = \overline{m+1})$$

Let a be arbitrary. By Q_3, either $a = \mathsf{o}$ or for some b, $a = b'$. In the first case, we have $\overline{m}' + a = \overline{m+1}$ by Q_4, and so $a < \overline{m+1}$ by Q_8.

Now consider the second case, $a = b'$. By the induction hypothesis, $(b < \overline{m} \lor \overline{m} < b) \lor b = \overline{m}$.

The first disjunct $b < \overline{m}$ is equivalent (by Q_8) to $\exists z\,(z' + b) = \overline{m}$. Suppose c has this property. If $(c' + b) = \overline{m}$, then also $(c' + b)' = \overline{m}'$. By Q_5, $(c' + b)' = (c' + b')$. Hence, $(c' + b') = \overline{m}'$. We get $\exists u\,(u' + b') = \overline{m+1}$ by existentially generalizing on c' and keeping in mind that $\overline{m}' \equiv \overline{m+1}$. Hence, if $b < \overline{m}$ then $b' < \overline{m+1}$ and so $a < \overline{m+1}$.

Now suppose $\overline{m} < b$, i.e., $\exists z\,(z' + \overline{m}) = b$. Suppose c is such a z, i.e., $(c' + \overline{m}) = b$. By logic, $(c' + \overline{m})' = b'$. By Q_5, $(c' + \overline{m}') = b'$. Since $a = b'$ and $\overline{m}' \equiv \overline{m+1}$, $(c' + \overline{m+1}) = a$. By Q_8, $\overline{m+1} < a$.

Finally, assume $b = \overline{m}$. Then, by logic, $b' = \overline{m}'$, and so $a = \overline{m+1}$.

Hence, from each disjunct of the case for m and b, we can obtain the corresponding disjunct for for $m+1$ and a. □

inc:req:min: prop:rep-minimization

Proposition req.5. *If $\varphi_g(x, z, y)$ represents $g(x, y)$ in* **Q***, then*

$$\varphi_f(z, y) \equiv \varphi_g(y, z, \mathsf{o}) \land \forall w\,(w < y \to \lnot\varphi_g(w, z, \mathsf{o})).$$

represents $f(z) = \mu x\,[g(x, z) = 0]$.

Proof. First we show that if $f(n) = m$, then $\mathbf{Q} \vdash \varphi_f(\overline{n}, \overline{m})$, i.e.,

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, \mathsf{o}) \land \forall w\,(w < \overline{m} \to \lnot\varphi_g(w, \overline{n}, \mathsf{o})).$$

Since $\varphi_g(x, z, y)$ represents $g(x, z)$ and $g(m, n) = 0$ if $f(n) = m$, we have

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, \mathsf{o}).$$

If $f(n) = m$, then for every $k < m$, $g(k, n) \neq 0$. So

$$\mathbf{Q} \vdash \lnot\varphi_g(\overline{k}, \overline{n}, \mathsf{o}).$$

We get that

inc:req:min: rep-less

$$\mathbf{Q} \vdash \forall w\,(w < \overline{m} \to \lnot\varphi_g(w, \overline{n}, \mathsf{o})). \tag{6}$$

by Lemma req.2 in case $m = 0$ and by Lemma req.3 otherwise.

Now let's show that if $f(n) = m$, then $\mathbf{Q} \vdash \forall y\,(\varphi_f(\overline{n}, y) \to y = \overline{m})$. We again sketch the argument informally, leaving the formalization to the reader.

Suppose $\varphi_f(\overline{n}, b)$. From this we get (a) $\varphi_g(b, \overline{n}, \mathsf{o})$ and (b) $\forall w\,(w < b \to \lnot\varphi_g(w, \overline{n}, \mathsf{o}))$. By Lemma req.4, $(b < \overline{m} \lor \overline{m} < b) \lor b = \overline{m}$. We'll show that both $b < \overline{m}$ and $\overline{m} < b$ leads to a contradiction.

If $\overline{m} < b$, then $\lnot\varphi_g(\overline{m}, \overline{n}, \mathsf{o})$ from (b). But $m = f(n)$, so $g(m, n) = 0$, and so $\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, \mathsf{o})$ since φ_g represents g. So we have a contradiction.

Now suppose $b < \overline{m}$. Then since $\mathbf{Q} \vdash \forall w\,(w < \overline{m} \to \lnot\varphi_g(w, \overline{n}, \mathsf{o}))$ by eq. (6), we get $\lnot\varphi_g(b, \overline{n}, \mathsf{o})$. This again contradicts (a). □

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Bibliography