

req.1 Composition is Representable in Q

inc:req:cmp:
sec Suppose h is defined by

$$h(x_0, \dots, x_{l-1}) = f(g_0(x_0, \dots, x_{l-1}), \dots, g_{k-1}(x_0, \dots, x_{l-1})).$$

where we have already found **formulas** $\varphi_f, \varphi_{g_0}, \dots, \varphi_{g_{k-1}}$ representing the functions f , and g_0, \dots, g_{k-1} , respectively. We have to find **a formula** φ_h representing h .

Let's start with a simple case, where all functions are 1-place, i.e., consider $h(x) = f(g(x))$. If $\varphi_f(y, z)$ represents f , and $\varphi_g(x, y)$ represents g , we need **a formula** $\varphi_h(x, z)$ that represents h . Note that $h(x) = z$ iff there is a y such that both $z = f(y)$ and $y = g(x)$. (If $h(x) = z$, then $g(x)$ is such a y ; if such a y exists, then since $y = g(x)$ and $z = f(y)$, $z = f(g(x))$.) This suggests that $\exists y (\varphi_g(x, y) \wedge \varphi_f(y, z))$ is a good candidate for $\varphi_h(x, z)$. We just have to verify that **Q** proves the relevant **formulas**.

inc:req:cmp:
prop:rep1 **Proposition req.1.** *If $h(n) = m$, then $\mathbf{Q} \vdash \varphi_h(\bar{n}, \bar{m})$.*

Proof. Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$\mathbf{Q} \vdash \varphi_g(\bar{n}, \bar{k})$$

since φ_g represents g , and

$$\mathbf{Q} \vdash \varphi_f(\bar{k}, \bar{m})$$

since φ_f represents f . Thus,

$$\mathbf{Q} \vdash \varphi_g(\bar{n}, \bar{k}) \wedge \varphi_f(\bar{k}, \bar{m})$$

and consequently also

$$\mathbf{Q} \vdash \exists y (\varphi_g(\bar{n}, y) \wedge \varphi_f(y, \bar{m})),$$

i.e., $\mathbf{Q} \vdash \varphi_h(\bar{n}, \bar{m})$. □

inc:req:cmp:
prop:rep2 **Proposition req.2.** *If $h(n) = m$, then $\mathbf{Q} \vdash \forall z (\varphi_h(\bar{n}, z) \rightarrow z = \bar{m})$.*

Proof. Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$\mathbf{Q} \vdash \forall y (\varphi_g(\bar{n}, y) \rightarrow y = \bar{k})$$

since φ_g represents g , and

$$\mathbf{Q} \vdash \forall z (\varphi_f(\bar{k}, z) \rightarrow z = \bar{m})$$

since φ_f represents f . Using just a little bit of logic, we can show that also

$$\mathbf{Q} \vdash \forall z (\exists y (\varphi_g(\bar{n}, y) \wedge \varphi_f(y, z)) \rightarrow z = \bar{m}).$$

i.e., $\mathbf{Q} \vdash \forall y (\varphi_h(\bar{n}, y) \rightarrow y = \bar{m})$. □

The same idea works in the more complex case where f and g_i have arity greater than 1.

Proposition req.3. *If $\varphi_f(y_0, \dots, y_{k-1}, z)$ represents $f(y_0, \dots, y_{k-1})$ in \mathbf{Q} , inc:req:cmp:
prop:rep-composition and $\varphi_{g_i}(x_0, \dots, x_{l-1}, y)$ represents $g_i(x_0, \dots, x_{l-1})$ in \mathbf{Q} , then*

$$\exists y_0 \dots \exists y_{k-1} (\varphi_{g_0}(x_0, \dots, x_{l-1}, y_0) \wedge \dots \wedge \varphi_{g_{k-1}}(x_0, \dots, x_{l-1}, y_{k-1}) \wedge \varphi_f(y_0, \dots, y_{k-1}, z))$$

represents

$$h(x_0, \dots, x_{l-1}) = f(g_0(x_0, \dots, x_{l-1}), \dots, g_{k-1}(x_0, \dots, x_{l-1})).$$

Proof. Exercise. □

Problem req.1. Using the proofs of **Proposition req.2** and **Proposition req.2** as a guide, carry out the proof of **Proposition req.3** in detail.

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Bibliography