req.1 Computable Functions are Representable in Q

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Theorem req.1. *Every computable function is representable in* **Q**.

Proof. For definiteness, and using the Church–Turing Thesis, let's say that a function is computable iff it is general recursive. The general recursive functions are those which can be defined from the zero function zero, the successor function succ, and the projection function P_i^n using composition, primitive recursion, and regular minimization. By **??**, any function h that can be defined from f and g can also be defined using composition and regular minimization from f, g, and zero, succ, P_i^n, add, mult, $\chi_=$. Consequently, a function is general recursive iff it can be defined from zero, succ, P_i^n, add, mult, $\chi_=$ using composition and regular minimization.

We've furthermore shown that the basic functions in question are representable in **Q** (**????????????**), and that any function defined from representable functions by composition or regular minimization (**??**, **??**) is also representable. Thus every general recursive function is representable in **Q**. □

We have shown that the set of computable functions can be characterized as the set of functions representable in **Q**. In fact, the proof is more general. From the definition of representability, it is not hard to see that any theory extending **Q** (or in which one can interpret **Q**) can represent the computable functions. But, conversely, in any derivation system in which the notion of derivation is computable, every representable function is computable. So, for example, the set of computable functions can be characterized as the set of functions representable in Peano arithmetic, or even Zermelo–Fraenkel set theory. As Gödel noted, this is somewhat surprising. We will see that when it comes to provability, questions are very sensitive to which theory you consider; roughly, the stronger the axioms, the more you can prove. But across a wide range of axiomatic theories, the representable functions are exactly the computable ones; stronger theories do not represent more functions as long as they are axiomatizable. explanation

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Bibliography